LEARNING ALGEBRAIC DECOMPOSITIONS USING PRONY STRUCTURES

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Abstract. We propose an algebraic framework generalizing several variants of Prony's method and explaining their relations. This includes Hankel and Toeplitz variants of Prony's method for the decomposition of multivariate exponential sums, polynomials (w.r.t. the monomial and Chebyshev bases), Gaußian sums, spherical harmonic sums, taking also into account whether they have their support on an algebraic set.

Introduction

Learning decompositions of functions from their evaluations in terms of a given basis and similar questions like the moment problem are fundamental tasks in signal processing and related areas.

In 1795, Prony proposed an algebraic approach to give an answer to such a question in the case of univariate exponential sums [45]. Classic applications of Prony's method include for example Sylvester's method for Waring decompositions of binary forms [53, 54] and Padé approximation [57]. Since then these tools have been further developed [41, 37, 43, 52], new applications have been found (see, e.g., [26, Section 2.2] for connections to the Berlekamp-Massey algorithm), and recently also advances have been made on multivariate versions. Direct attempts can be found in, e.g., [42, 38, 31, 46, 36], for methods based on projections to univariate exponential sums see, e.g., [14, 15, 12]. A numerical variant can be found in, e.g., [17], and further related results and applications in, e.g., [16, 20, 32, 10, 27, 44, 7, 8, 25, 23].

As important as (approximate) algorithms undeniably are in practice, at its core Prony's method is of a purely algebraic nature which is the point of view of this article. We introduce a general algebraic framework called Prony structures for reconstruction methods modeled after Prony's original idea. Our approach allows a simultaneous treatment of decomposition problems in particular for multivariate exponential sums, polynomials (w.r.t. the monomial and Chebyshev bases), Gaußian sums, and eigenvector sums of linear operators.

To describe the main task, consider a vector space V of functions with a distinguished basis B. The goal is to decompose an arbitrary function $f \in V$ into a linear combination of basis elements. As a constraint for this it is only allowed to use evaluations of f.

In typical Prony situations one has a way to identify basis elements with points in an affine space. For example, in the case of exponential sums the basis function $\exp_b$ is identified with its base point $b \in \mathbb{C}^n$. It is this identification that allows to describe the support of f, i.e. the used basis elements in the decomposition, by polynomial equations.

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A key idea of Prony is to construct Hankel (or Toeplitz) matrices using evaluations of f to obtain the desired data from their kernels.

In our framework we assume that an identification as above is given as part of the initial data. Then suitable sequences of matrices are computed from evaluations of f which are constructed in a way such that their kernels eventually have to yield systems of polynomial equations to determine the support of f.

The article is organized as follows. In Section 1 we fix the setup, some notation, and introduce our main definition of a Prony structure. Besides the function space and the basis as key parts of the data it consists of families of matrices and associated ideals defined by their kernels. These ideals are then used to attack the decomposition problem. We also recall briefly, as a special case, the fundamental example of Prony's classic method.

In Section 2 we discuss properties of evaluation maps on vector spaces of polynomials and their kernels, see for example [31]. As one of our main results, we prove in Theorem 2.4 a very useful characterization of Prony structures in terms of factorizations through evaluation maps.

It can be seen that given some mild assumptions the ideals of a Prony structure are zero-dimensional and radical (see Corollary 3.2), which leads to the natural question to provide sufficient conditions which guarantee that the ideals of kernels of evaluation maps have this property.

In Section 3 we study this problem. The main result of this section (Theorem 3.10) proves a theorem of Möller on Gröbner bases of zero-dimensional radical ideals with interesting consequences for Prony structures.

In Section 4 we discuss in particular in Theorem 4.4 fundamental examples of Prony structures based on the Hankel and Toeplitz matrices defined by exponential sums, see for example [36] for their use in classic situations related to Prony's methods.

Known reconstruction techniques can be used for sums of eigenvector of linear operators [37], polynomials (w.r.t. the monomial and various types of Chebyshev bases) [4, 32, 27, 44, 36], and multivariate Gaußians [38]. In Section 5 we will see in particular that they arise from Prony structures related to those for exponential sums. In this section we also show relations between the framework of Prony structures and previously known frameworks for character [16] and eigenfunction sums [20, 37].

A priori knowledge can be that functions are supported for example on a torus or a sphere, see, e.g., [29, 30]. Classic techniques do not take this additional information into account. As a novel approach we extend the notion of Prony structures for functions supported on algebraic sets to a relative version in Section 6. A first key result is a characterization of such structures in Theorem 6.8. In Theorem 6.9 and its corollaries we discuss how to obtain Prony structures in this relative case. Main examples include relative Prony structures for spaces of spherical harmonics.

Already in the existing literature, projection techniques are used to apply Prony's method, see, e.g. [14, 15, 12]. Related to this idea is an observation in Section 5 that a Prony structure may be "induced" by another one on a different vector space. The systematic point of view of these phenomena is given by maps between Prony structures, which we introduce in Section 7. We discuss projection methods, Gaußian sums and other examples in terms of such maps.

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1. Prony structures

Motivated by Prony's reconstruction method as well as its recent generalizations we introduce a framework that enables us to treat several of these variants simultaneously and which can be applied in various contexts. In this section we begin by fixing some notation regarding evaluation maps for polynomials, and then make our main definition of Prony structures. The key point is to give a general formal setting that captures the essence of Prony's method with the aim of laying the foundation for a structural theory.

Definition 1.1. Let K be a field, $n \in \mathbb{N}$, $S := K[\mathrm{x}] := K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, and for an arbitrary subset $D \subseteq \mathbb{N}^n$ let $S_D := \langle \mathrm{x}^D \rangle_K = \langle \mathrm{x}^\alpha \mid \alpha \in D \rangle_K$. For $X \subseteq K^n$ define

$$\mathrm{ev}_D^X \colon S_D \to K^X, \quad p \mapsto (p(x))_{x \in X},$$

and

$$\mathrm{I}_D(X) := \ker(\mathrm{ev}_D^X).$$

We call ev_D^X the *evaluation map at* X and $\mathrm{I}_D(X)$ the *vanishing space of* X w.r.t. S_D.

Observe that for $D = \mathbb{N}^n$ we just have $S_D = S$ and $\mathrm{I}(X) = \mathrm{I}_{\mathbb{N}^n}(X)$ is the usual *vanishing ideal of* X. In this special situation we also set $\mathrm{ev}^X := \mathrm{ev}_{\mathbb{N}^n}^X$. Note that in general we have

$$\mathrm{I}_D(X) = \mathrm{I}(X) \cap S_D.$$

In Section 2 we will state all results on evaluation maps and their kernels that are relevant for this note.

In order to characterize basis elements of a vector space V through systems of polynomial equations we need a way to identify them with points. This will be achieved by an injection u as in the following definition.

Definition 1.2. Let F be a field, V be an F-vector space, and B be an F-basis of V. For $f \in V$, $f = \sum_{i=1}^r f_i b_i$ with $f_1, \ldots, f_r \in F \setminus \{0\}$ and distinct $b_1, \ldots, b_r \in B$, let

$$\mathrm{supp}_B(f) := \{b_1, \ldots, b_r\} \quad \text{and} \quad \mathrm{rank}_B(f) := |\mathrm{supp}_B(f)| = r$$

denote the *support of* f and *rank of* f (w.r.t. B), respectively. For a field K, $n \in \mathbb{N}$, and an injective map $u \colon B \to K^n$ let

$$\mathrm{supp}_u(f) := \{u(b_1), \ldots, u(b_r)\}.$$

We call $\mathrm{supp}_u(f) \subseteq K^n$ the *u-support* and its elements the *support labels of* f.

In many situations we will choose $K = F$, but for reasons of flexibility we allow the choice of possibly different fields. Unless mentioned otherwise, we will assume that F, V, B, K, n, and u are given as in Definition 1.2. In the following definition we introduce the central notion of a Prony structure.

Definition 1.3. Given the setup of Definition 1.2, let $\mathcal{I} = (\mathcal{I}_d)_{d \in \mathbb{N}}$ be a sequence of finite sets and $\mathcal{J} = (\mathcal{J}_d)_{d \in \mathbb{N}}$ be a sequence of finite subsets of $\mathbb{N}^n$. Let $f \in V$ and

$$P(f) = (P_d(f))_{d \in \mathbb{N}} \in \prod_{d \in \mathbb{N}} K^{\mathcal{I}_d \times \mathcal{J}_d},$$

i.e., a family of matrices with $P_d(f) \in K^{\mathcal{I}_d \times \mathcal{J}_d}$ for all $d \in \mathbb{N}$. We call $P(f)$ a *Prony structure for* f if there is a $c \in \mathbb{N}$ such that for all $d \in \mathbb{N}$ with $d \geq c$ one has

(1) $$\mathrm{Z}(\ker P_d(f)) = \mathrm{supp}_u(f) \quad \text{and} \quad \mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f)) \subseteq \ker(P_d(f)).[1]$$

[1]See Remark 1.8 for a discussion of the second condition, which is not implied by the other.

Here we identify $p \in \ker P_d(f) \subseteq K^{\mathcal{J}_d}$ with the polynomial $\sum_{\alpha\in\mathcal{J}_d} p_\alpha \mathrm{x}^\alpha \in K[\mathrm{x}_1,\dots,\mathrm{x}_n]$ and $\mathrm{Z}(\cdot)$ takes the zero locus of a set of polynomials.

The least $c \in \mathbb{N}$ such that the conditions in (1) hold for all $d \geq c$ is called *Prony index of f* or simply *P-index of f*, denoted by $\operatorname{ind}_P(f)$.

If for every $f \in V$ a Prony structure $P(f)$ for f is given, then we call P a *Prony structure on V*.

Remark 1.4. A key point of a Prony structure P on V is that the idea of Prony's method works, i.e. to compute the support of a given $f \in V$ w.r.t. the basis B through a system of polynomial equations. More precisely, one can perform the following (pseudo-)algorithm:

(1) Choose $d \in \mathbb{N}$.
(2) Determine $P_d(f) \in K^{\mathcal{I}_d \times \mathcal{J}_d}$.
(3) Compute $U := \ker P_d(f) \subseteq K^{\mathcal{J}_d}$.
(4) Embed $U \subseteq K[\mathrm{x}_1,\dots,\mathrm{x}_n]$.
(5) Compute $Z := \mathrm{Z}(U) \subseteq K^n$.
(6) Compute $u^{-1}(Z) \subseteq B$.

If d is chosen large enough, then the zero locus Z is the u-support and $u^{-1}(Z)$ is the support of f (and in particular these sets are finite). Note that for this strategy to work it is important that the matrices $P_d(f)$ can be determined from "standard information" on f (such as evaluations if f is a function), in particular without already knowing the support; see also Remark 1.6. Often computation of the zero locus as well as a good choice of d turn out to be problematic steps.

In classic situations of Prony's method the non-zero coefficients of f w.r.t. B can be computed in an additional step by solving a system of linear equations involving only standard information; this system is finite since one has already computed the support. We omit the discussion of this step here and in the following.

Common options for the sequence $\mathcal{J} = (\mathcal{J}_d)_{d\in\mathbb{N}}$ are $\mathcal{J} = \mathcal{T}$, $\mathcal{J} = \mathcal{M}$, or $\mathcal{J} = \mathcal{C}$, where

$$\begin{aligned}
\mathcal{T}_d &:= \Big\{\alpha \in \mathbb{N}^n \;\Big|\; \sum_{j=1}^n \alpha_j \leq d\Big\},\\
\mathcal{M}_d &:= \{\alpha \in \mathbb{N}^n \mid \max\{\alpha_j \mid j = 1,\dots,n\} \leq d\},\\
\text{and}\quad \mathcal{C}_d &:= \Big\{\alpha \in \mathbb{N}^n \;\Big|\; \prod_{j=1}^n (\alpha_j + 1) \leq d\Big\}.
\end{aligned}$$

Under the identification of $K^{\mathcal{J}_d}$ with polynomials, in $S = K[\mathrm{x}_1,\dots,\mathrm{x}_n]$ the choice $\mathcal{J} = \mathcal{T}$ corresponds to the subvector space of polynomials of *total degree* at most d, and $\mathcal{J} = \mathcal{M}$ corresponds to the subvector space of polynomials of *maximal degree* at most d. Choosing $\mathcal{J} = \mathcal{C}_d$, the non-negative orthant of the *hyperbolic cross* of order d, gives rise to a space of polynomials that is particularly well-suited for zero-testing and interpolation of polynomials. The earliest use of $\mathcal{C}_d$ in the context of Prony-like methods that we are aware of is in articles by Clausen, Dress, Grabmeier, and Karpinski [6] and by Dress and Grabmeier [16]. For more recent applications see in particular Sauer [47] and the preprint Hubert-Singer [23].

Often one chooses $\mathcal{I} = \mathcal{J}$, $\mathcal{I}_d = \mathcal{J}_{d-1}$, or a similar relation between $\mathcal{I}$ and $\mathcal{J}$.

We will also use the notation

$$S_{\leq d} := S_{\mathcal{T}_d} = \langle \mathrm{x}^\alpha \mid \alpha \in \mathcal{T}_d\rangle_K, \quad \mathrm{ev}^X_{\leq d} := \mathrm{ev}^X_{\mathcal{T}_d}, \quad \text{and} \quad \mathrm{I}_{\leq d}(X) := \mathrm{I}_{\mathcal{T}_d}(X).$$

Remark 1.5. A framework for the decomposition of sums of characters of commutative monoids has been proposed in Dress-Grabmeier [16] and derivations for sums of eigenfunctions (or more generally eigenvectors) of linear operators have been developed in Grigoriev-Karpinski-Singer [20] and Peter-Plonka [37]. We recast these frameworks in the language of Prony structures in Section 5. See Remark 5.16 for a diagrammatic overview.

While there is considerable overlap with the one proposed here, the two approaches make different compromises between generality and effectivity. We aim at a formalization of the most general situation in which Prony's strategy still works. Our treatment is axiomatic rather than the explicit constructions of [16, 20, 37]. While trading in some directness, this abstraction also allows to stay within the language of linear algebra. When dealing with applications, a detour through character sums can seem unnatural (or, as in the Chebyshev decomposition, impossible) given the concrete situation. In this sense, we also find our framework to be more effectively verifiable.

Remark 1.6. For $f \in V$ let $P_d(f)$ denote the matrix of $\mathrm{ev}_{\leq d}^{\mathrm{supp}_u(f)}$ w.r.t. the monomial basis of $K[\mathrm{x}]_{\leq d}$ and the canonical basis of $K^{\mathrm{supp}_u(f)}$. Then $(P_d(f))_{d\in\mathbb{N}}$ is a Prony structure for f, cf. Lemma 2.1.

For practical computation of the support of f this Prony structure is useless, since clearly $P_d(f)$ is the Vandermonde-like matrix $\mathrm{V}_{\mathcal{T}_d}^{\mathrm{supp}_u(f)} = (x^\alpha)_{x\in\mathrm{supp}_u(f),|\alpha|\leq d}$ and knowing these matrices immediately implies knowledge of the u-support of f. This observation does however provide a possible strategy to construct Prony structures that may be obtained from some available data, see Corollary 2.2.

We recall the classic Prony's method for reconstructing univariate exponential sums, which dates back to 1795 [45]. It is the fundamental example of a Prony structure.

Example 1.7. For $b \in \mathbb{C}$ we call the function

$$\exp_b\colon \mathbb{N} \to \mathbb{C}, \quad \alpha \mapsto b^\alpha,$$

exponential (*with base* b) and we call $\mathbb{C}$-linear combinations of exponentials *exponential sums*. Here it is understood that $0^0 = 1$. We denote by $B := \{\exp_b \mid b \in \mathbb{C}\}$ the set of all exponentials, which is a $\mathbb{C}$-basis of the vector space

$$V := \mathrm{Exp} := \langle B\rangle_{\mathbb{C}} = \{f\colon \mathbb{N} \to \mathbb{C} \mid f \text{ exponential sum}\}.$$

Then the classic Prony problem is to determine the coefficients $f_i \in \mathbb{C} \setminus \{0\}$ and the bases $b_i \in \mathbb{C}$ of a given exponential sum $f = \sum_{i=1}^r f_i \exp_{b_i} \in \mathrm{Exp}$. Of course, the function

$$u\colon B \to \mathbb{C}, \quad \exp_b \mapsto b = \exp_b(1),$$

is a bijection. For an exponential sum $f \in \mathrm{Exp}$ and $d \in \mathbb{N}$, consider the Hankel matrix

$$\mathrm{H}_d(f) := (f(\alpha+\beta))_{\substack{\alpha=0,\ldots,d-1\\ \beta=0,\ldots,d}} = \begin{pmatrix} f(0) & f(1) & \cdots & f(d) \\ f(1) & f(2) & \cdots & f(d+1) \\ \vdots & \vdots & \vdots & \vdots \\ f(d-1) & f(d) & \cdots & f(2d-1) \end{pmatrix} \in \mathbb{C}^{d\times(d+1)}.$$

Prony has shown in his 1795 *Essai* [45] that H is a Prony structure on Exp and, moreover, for every $f \in \mathrm{Exp}$, $\mathrm{ind}_{\mathrm{H}}(f) = \mathrm{rank}_B(f)$. This provides a method to compute $\mathrm{supp}_u(f) = \mathrm{Z}(\ker \mathrm{H}_d(f)) \subseteq \mathbb{C}$, under the assumption that an upper bound $d = d_f \in \mathbb{N}$ of $\mathrm{rank}_B(f)$ is known. (Multivariate) generalizations and variants of Prony's method will be discussed in Sections 4 and 5 (see also Peter-Plonka [37], Kunis-Peter-Römer-von der Ohe [31], Sauer [46], and Mourrain [36]).

Remark 1.8. One might be tempted to remove the technical *vanishing space condition*

$$\mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f)) \subseteq \ker(P_d(f))$$

from Definition 1.3. For the sake of discussion, call P a *quasi Prony structure for f* if P satisfies all the conditions of a Prony structure for f in Definition 1.3 with the only possible exception of the vanishing space condition. We observe the following:

(a) All practically relevant examples of quasi Prony structures that we are aware of are indeed Prony structures.

(b) One of the main reasons why we include the vanishing space condition in the definition of Prony structures is that the analogues of several of our statements on Prony structures do not hold or are not known to hold for quasi Prony structures; see, for example, Theorem 2.4 and Theorem 6.9.

(c) An "artificial" example of a quasi Prony structure that is not a Prony structure: For $d \in \mathbb{N}$ let $\mathcal{I}_d = \{0,1\}$, $\mathcal{J}_d = \{0,1,2\}$, and $P_d := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbb{C}^{\mathcal{I}_d \times \mathcal{J}_d}$. Then $\ker(P_d) = \langle \mathrm{x}^2 \rangle_{\mathbb{C}} \subseteq \mathbb{C}[\mathrm{x}]$, so $\mathrm{Z}(\ker(P_d)) = \mathrm{Z}(\mathrm{x}^2) = \{0\}$, hence $P = (P_d)_{d\in\mathbb{N}}$ is a quasi Prony structure for $f := \exp_0 \in \mathrm{Exp}$ (cf. Example 1.7). Since $\mathrm{x} \in \mathrm{I}_{\mathcal{J}_d}(0) \setminus \ker(P_d)$ for all d, P is not a Prony structure for f.

Remark 1.9.

(a) The generalization of Prony's problem to polynomial-exponential sums (sums of functions $\alpha \mapsto p(\alpha)\exp_b(\alpha)$ with polynomials p), also known as "multiplicity case", can be found in the univariate case in Henrici [21, Theorem 7.2 c]. Further developments such as a characterization of sequences that allow interpolation by polynomial-exponential sums have been obtained by Sidi [51] and a variant based on an associated generalized eigenvalue problem is given in Lee [33], see also Peter-Plonka [37, Theorem 2.4] and Stampfer-Plonka [52]. For generalizations of many of these results to the multivariate setting see Mourrain [36]. It would be interesting to extend the notion of Prony structures to also include these cases. We leave this for future work. See also Remark 5.6.

(b) In general, if $P(f)$ is a Prony structure for f and K is algebraically closed, then, for all $d \geq \mathrm{ind}_P(f)$, we have $\mathrm{rad}(\langle \ker P_d(f) \rangle) = \mathrm{I}(\mathrm{Z}(\ker P_d(f))) = \mathrm{I}(\mathrm{supp}_u(f))$ by Hilbert's Nullstellensatz. It is an interesting problem whether always or under which conditions the ideal $\langle \ker P_d(f) \rangle$ is already a radical ideal. We return to this question in Section 3 where we provide partial answers also over not necessarily algebraically closed fields.

2. Prony structures and the evaluation map

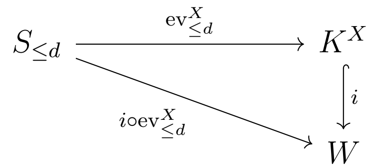
In this section we recall some well-known properties of evaluation maps on vector spaces of polynomials and their kernels. Since they are the vector spaces of polynomials vanishing on a set $X \subseteq K^n$, these kernels play a crucial role in the theory and application of Prony structures, which will be made precise in Theorem 2.4.

We provide in this section the essential facts. Related issues will be studied in more detail in Section 3.

Lemma 2.1. *Let $X \subseteq K^n$. Then there is a $d \in \mathbb{N}$ with $\langle \mathrm{I}_{\leq d}(X) \rangle = \mathrm{I}(X)$. For finite X this implies $\mathrm{Z}(\mathrm{I}_{\leq d}(X)) = X$.*

Proof. This follows immediately from the fact that $S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ is Noetherian and thus $\mathrm{I}(X)$ is finitely generated for $X \subseteq K^n$. If X is finite, then it is Zariski closed. □

Corollary 2.2. *Let $X \subseteq K^n$ be finite. Then for any K-vector space W and injective K-linear map $i\colon K^X \hookrightarrow W$ one has $\mathrm{I}_{\leq d}(X) = \ker(i \circ \mathrm{ev}^X_{\leq d})$. In particular, $\mathrm{Z}(\ker(i \circ \mathrm{ev}^X_{\leq d})) = X$ for all large d. The following diagram illustrates the situation.*



Proof. The first statement clearly holds and the second one follows from Lemma 2.1. ☐

The following result on polynomial interpolation is well-known.

Lemma 2.3. *Let $X \subseteq K^n$ be finite. If $d \in \mathbb{N}$ and $d \geq |X| - 1$ then $\mathrm{ev}^X_{\leq d}$ is surjective.*

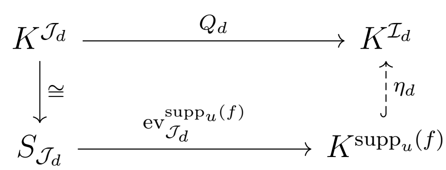
Proof. It is easy to see that given $x \in X$, there is a polynomial $p \in S$ of degree $|X|-1$ such that $p(x) = 1$ and $p(y) = 0$ for $y \in X \setminus \{x\}$ (see, e.g., the proof of Cox-Little-O'Shea [9, Chapter 5, §3, Proposition 7]). By linearity this concludes the proof. ☐

As the main result of this section we obtain the following characterization of Prony structures.

Theorem 2.4. *Given the setup of Definition 1.2, let $f \in V$, B an F-basis of V, $u\colon B \to K^n$ injective, $\mathcal{I}$ a sequence of finite sets, and $\mathcal{J}$ a sequence of finite subsets of $\mathbb{N}^n$ with $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ for all large d and $\bigcup_{d \in \mathbb{N}} \mathcal{J}_d = \mathbb{N}^n$. Let $Q \in \prod_{d \in \mathbb{N}} K^{\mathcal{I}_d \times \mathcal{J}_d}$. Then the following are equivalent:*

(i) *Q is a Prony structure for f;*

(ii) *For all large d there is an injective K-linear map $\eta_d\colon K^{\mathrm{supp}_u(f)} \hookrightarrow K^{\mathcal{I}_d}$ such that the diagram*



is commutative;

(iii) *For all large d we have $\ker(Q_d) = \mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f))$.*

Proof. (i) ⇒ (ii): By Definition 1.1 and since Q is a Prony structure for f, for all large d we have

$$\ker(\mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d}) = \mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f)) \subseteq \ker(Q_d).$$

By the hypotheses on $\mathcal{J}$, $\mathcal{T}_{|\mathrm{supp}_u(f)|} \subseteq \mathcal{J}_d$ for all large d. Then $\mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d}$ is surjective by Lemma 2.3. Together, these facts imply the existence of K-linear maps η_d such that the required diagrams are commutative.

It remains to show that η_d is injective for all large d. Let $c \in \mathbb{N}$ be such that for all $d \geq c$ we have that

$$\mathrm{Z}(\ker Q_d) = \mathrm{supp}_u(f), \quad \mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d} \text{ is surjective,} \quad \text{and } \eta_d \text{ exists.}$$

Let $v \in \ker(\eta_d)$. By surjectivity of $\mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d}$ we have $\mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d}(p) = v$ for some $p \in S_{\mathcal{J}_d}$. Then $Q_d(p) = \eta_d(\mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{J}_d}(p)) = \eta_d(v) = 0$. Thus, we have

$$p \in \ker(Q_d) \subseteq \mathrm{I}(\mathrm{Z}(\ker Q_d)) = \mathrm{I}(\mathrm{supp}_u(f)) = \ker(\mathrm{ev}^{\mathrm{supp}_u(f)}).$$

Hence, $v = \mathrm{ev}^{\mathrm{supp}_u(f)}(p) = 0$. Thus, η_d is injective.

(ii) ⇒ (iii): Since η_d exists and is injective (for all large d), we have

$$\ker(Q_d) = \ker(\eta_d \circ \mathrm{ev}_{\mathcal{J}_d}^{\mathrm{supp}_u(f)}) = \ker(\mathrm{ev}_{\mathcal{J}_d}^{\mathrm{supp}_u(f)}) = \mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f)).$$

(iii) ⇒ (i): By our hypothesis and Lemma 2.1, for all large d we have

$$\mathrm{Z}(\ker Q_d) = \mathrm{Z}(\mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f))) = \mathrm{Z}(\mathrm{I}(\mathrm{supp}_u(f))) = \mathrm{supp}_u(f).$$

The vanishing space condition in Definition 1.3 (1) is obviously satisfied. □

The art of constructing a "computable" Prony structure for a given $f \in V$ and the very heart of Prony's method is to find an injective K-linear map $\eta_d\colon K^{\mathrm{supp}_u(f)} \hookrightarrow W_d$ into a K-vector space W_d such that (a matrix of) the composition

$$P_d(f) := \eta_d \circ \mathrm{ev}_{\leq d}^{\mathrm{supp}_u(f)}\colon S_{\leq d} \to W_d$$

can be computed from standard data of f.

Remark 2.5. Let B be a generating subset of V. One can formulate a variation of Theorem 2.4 insofar that if one of the conditions (ii) or (iii) holds for all $f \in V$ and all representations $f = \sum_{b \in M} f_b b$ with $M \subseteq B$ finite and $f_b \in F \setminus \{0\}$, and replacing each occurrence of $\mathrm{supp}_u(f)$ by $u(M)$, then B is a basis of V and $Q_d(f)$ induces a Prony structure on V. Indeed, $\mathrm{Z}(\ker Q_d(f)) = M$ implies that M is uniquely determined by $Q_d(f)$, which implies the desired conclusion.

The following Proposition 2.6 (a) is a version of Lemma 2.1 that provides the upper bound $d = |X|$ for the "stabilization index" of the ascending sequence of ideals $(\langle \mathrm{I}_{\leq d}(X)\rangle)_{d\in\mathbb{N}}$. In part (b) it is shown that $|X| - 1$ is not in general an upper bound.

Proposition 2.6. *The following holds:*

(a) *Let $X \subseteq K^n$ be finite. With $d := |X|$ we have*

$$\langle \mathrm{I}_{\leq d}(X)\rangle = \mathrm{I}(X).$$

(b) *Let K be an infinite field. Then for every $d \in \mathbb{N}$ there is an $X \subseteq K^n$ with $|X| = d + 1$ such that $\langle \mathrm{I}_{\leq d}(X)\rangle \subsetneqq \mathrm{I}(X)$.*

Proof. (a) This is part of the proof of Kunis-Peter-Römer-von der Ohe [31, Theorem 3.1].

(b) Let $d \in \mathbb{N}$. Since K is infinite, there exists

$$X = \{(x, 0, \ldots, 0) \in K^n \mid x \in X_1\} \text{ for some } X_1 \subseteq K \text{ with } |X| = |X_1| = d + 1.$$

Let $\mathrm{I}(X) = \langle E\rangle$ for some $E \subseteq S$. We claim that there is a $p \in E$ with $\deg(p) > d$.

For $p \in S$, let $\widetilde{p} := p(\mathrm{x}_1, 0, \ldots, 0)$. Assume $\widetilde{p} = 0$ for all $p \in E$. For $y \in K$ we have $(y, 0, \ldots, 0) \in \mathrm{Z}(E) = \mathrm{Z}(\mathrm{I}(X)) = X$ and hence $y \in X_1$. We get the contradiction $X_1 = K$.

Thus there is a $p \in E$ with $\widetilde{p} \neq 0$. Since $\widetilde{p}(x) = 0$ for $x \in X_1$ and $|X_1| = d + 1$, we have

$$\deg(p) \geq \deg(\widetilde{p}) \geq d + 1.$$

This concludes the proof. □

3. Properties of the evaluation map and a theorem of Möller

Continuing the discussion in Section 2 we study in the following further properties of evaluation maps and we provide partial answers to the question raised in Remark 1.9 (b).

This section is to some degree independent from the rest of the article. The reader who wishes to continue directly with applications of Prony structures and is not particularly concerned with the ideal-theoretic issues treated here can safely skip this section.

The consequences of the results of this section for Prony structures are summarized in Corollary 3.12.

We are grateful towards H. M. Möller for inspiring discussions related to these results, in particular for allowing us to include Theorem 3.10 and its proof [35].

As before, let $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$ be the polynomial ring in n indeterminates over the field K. In the following we do not distinguish between $\alpha \in \mathbb{N}^n$ and the monomial $\mathrm{x}^\alpha \in \mathrm{Mon}(S)$. For general facts about initial ideals and Gröbner bases see, e.g., Cox-Little-O'Shea [9].

Remark 3.1. Let $X \subseteq K^n$ be finite. A direct consequence of Proposition 2.6 (a) is that for all $d \geq |X|$ the vanishing spaces $\mathrm{I}_{\mathcal{T}_d}(X)$ generate the same radical ideal in S (namely, $\mathrm{I}(X)$).

As a consequence we get immediately:

Corollary 3.2. *Given the setup of Definition* 1.3*, let $P(f)$ be a Prony structure for $f \in V$ with $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ for all large d and $\bigcup_{d\in\mathbb{N}} \mathcal{J}_d = \mathbb{N}^n$. Then for all large d*

$$\langle \ker P_d(f) \rangle = \mathrm{I}(\mathrm{supp}_u(f)).$$

In particular, for all large d, $\langle \ker P_d(f) \rangle$ is a radical ideal in S.

Proof. Let $X := \mathrm{supp}_u(f)$ and $r := \mathrm{rank}(f) = |X|$. For all large d we have $\mathcal{T}_r \subseteq \mathcal{J}_d$ and $\ker P_d(f) = \mathrm{I}_{\mathcal{J}_d}(X) \supseteq \mathrm{I}_{\mathcal{T}_r}(X)$. Since also $\mathcal{J}_d \subseteq \mathcal{T}_e$ for an $e \in \mathbb{N}$, we have

$$\mathrm{I}(X) = \langle \mathrm{I}_{\mathcal{T}_r}(X) \rangle \subseteq \langle \mathrm{I}_{\mathcal{J}_d}(X) \rangle \subseteq \langle \mathrm{I}_{\mathcal{T}_e}(X) \rangle \subseteq \mathrm{I}(X).$$

This concludes the proof. ☐

Observe that $\langle \mathrm{I}_D(X) \rangle_S$ is not a radical ideal in general. This is shown already by the example $n = 1$, $X = \{0\}$, $D = \{\mathrm{x}_1^2\}$, where $\langle \mathrm{I}_D(X) \rangle = \langle \mathrm{x}_1^2 \rangle_S$. Note that for a given $X \subseteq K^n$, the map $\mathrm{ev}_{\leq d}^X$ can be surjective also for $d < |X| - 1$. Furthermore, $\mathrm{I}_{\leq d}(X)$ could also generate a radical ideal for small d. The following simple example illustrates this.

Example 3.3. Let $X := \{(0,0), (1,0), (0,1)\} \subseteq K^2$. One can see immediately that $\mathrm{ev}_{\leq 1}^X$ is bijective by considering its matrix

$$\mathrm{V}_{\leq 1}^X = (t(x))_{\substack{x\in X \\ t\in\mathcal{T}_1}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in K^{X \times \mathcal{T}_1}.$$

Therefore, $\mathrm{ev}_{\leq 1}^X$ is surjective and $\mathrm{I}_{\leq 1}(X) = \ker(\mathrm{ev}_{\leq 1}^X) = \{0\}$. So $\langle \mathrm{I}_{\leq 1}(X) \rangle$ is the zero ideal of S, which is prime and thus radical (and of course not equal to $\mathrm{I}(X)$). The vanishing ideal $\mathrm{I}(X)$ of X is generated by

$$\ker(\mathrm{ev}_{\leq 2}^X) = \langle \mathrm{x}_1(\mathrm{x}_1 - 1), \mathrm{x}_2(\mathrm{x}_2 - 1), \mathrm{x}_1\mathrm{x}_2 \rangle_K.$$

Having these facts in mind we consider special situations and prove results related to Corollary 3.2 and Example 3.3.

For a monomial order $<$ on $\mathrm{Mon}(S)$ and an ideal I of S we denote by

$$\mathrm{N}_<(I) := \mathrm{Mon}(S) \setminus \mathrm{in}_<(I)$$

the *normal set of I*. From now on we omit the monomial order from the notation and write, e.g., $\mathrm{in}(I)$ and $\mathrm{N}(I)$ for $\mathrm{in}_<(I)$ and $\mathrm{N}_<(I)$, respectively.

For example, for $I = \mathrm{I}(X)$ with $X \subseteq K^2$ as in Example 3.3, one has

$$\mathrm{in}(I) = \langle \mathrm{x}_1^2, \mathrm{x}_1\mathrm{x}_2, \mathrm{x}_2^2 \rangle \text{ and thus } \mathrm{N}(I) = \{1, \mathrm{x}_1, \mathrm{x}_2\}$$

for the degree reverse lexicographic order $<$.

Lemma 3.4. *Let $<$ be a monomial order on* $\operatorname{Mon}(S)$*,* $X \subseteq K^n$ *be finite and* $I := \operatorname{I}(X)$*. Then the following holds:*

(a) $\operatorname{ev}^X_{\operatorname{N}(I)} \colon S_{\operatorname{N}(I)} \to K^X$ *is bijective. In particular,* $|\operatorname{N}(I)| = |X|$.

(b) *Let* $D \subseteq \operatorname{Mon}(S)$ *be such that* $\operatorname{ev}^X_D \colon S_D \to K^X$ *is surjective. Then there is a* $C \subseteq \operatorname{Mon}(S)$ *with the following properties:*

(1) $C \subseteq D$.

(2) $\operatorname{ev}^X_C \colon S_C \to K^X$ *is bijective. In particular,* $|C| = |X| = |\operatorname{N}(I)|$.

(3) *For all* $t \in D \setminus C$ *we have* $\operatorname{ev}^X_D(t) \in \langle \operatorname{ev}^X_C(s) \mid s \in C \text{ and } s < t\rangle_K$.

Proof. (a) It is a standard fact that $S_{\operatorname{N}(I)} \cong S/I \cong K^X$, see, for example, Cox-Little-O'Shea [9, Chapter 5, §3, Proposition 4]. Let $p \in \ker(\operatorname{ev}^X_{\operatorname{N}(I)})$ and suppose that $p \neq 0$. Then $\operatorname{in}(p) \in \operatorname{in}(I) \cap \operatorname{N}(I) = \emptyset$, a contradiction. Thus, $\operatorname{ev}^X_{\operatorname{N}(I)}$ is injective and hence an isomorphism.

(b) Note that necessarily $|D| \geq |X|$. We prove the assertion by induction on $k = |D| - |X| \in \mathbb{N}$. If $k = 0$, then $|D| = |X|$. So ev^X_D is bijective and $C = D$ works trivially.

Let $k \geq 1$. Then $|D| > |X|$ and the elements $\operatorname{ev}^X_D(t)$, $t \in D$, are linearly dependent in K^X. Hence there are $\lambda_t \in K$ with $\sum_{t\in D} \lambda_t \operatorname{ev}^X_D(t) = 0$ and $\lambda_t \neq 0$ for at least one $t \in D$. Let

$$t_0 := \max\nolimits_<\{t \in D \mid \lambda_t \neq 0\} \text{ and } D_1 := D \setminus \{t_0\}.$$

Clearly, $\operatorname{ev}^X_{D_1} \colon S_{D_1} \to K^X$ is surjective and $|D_1| - |X| = k - 1$. By induction hypothesis there is a $C_1 \subseteq D_1$ such that

$$\operatorname{ev}^X_{C_1} \colon S_{C_1} \to K^X \text{ is bijective and } \operatorname{ev}^X_{D_1}(t) \in \langle \operatorname{ev}^X_{C_1}(s) \mid s \in C_1,\, s < t\rangle_K \text{ for all } t \in D_1 \setminus C_1.$$

Clearly, $C_1 \subseteq D$. We claim that $C := C_1$ fulfills the assertion also for D. It remains to show statement (3) for $t = t_0$. For this let $U := \langle \operatorname{ev}^X_C(s) \mid s \in C,\, s < t_0\rangle_K$. From the linear dependency above it follows that

$$\operatorname{ev}^X_D(t_0) = \sum_{s\in D_1} \mu_s \operatorname{ev}^X_{D_1}(s) = \sum_{s\in C} \mu_s \operatorname{ev}^X_C(s) + \sum_{s\in D_1\setminus C} \mu_s \operatorname{ev}^X_{D_1}(s) \text{ with } \mu_s \in K.$$

Trivially $\sum_{s\in C} \mu_s \operatorname{ev}^X_C(s) \in U$ since by the choice of t_0 we have $s < t_0$ for all $s \in D$ with $\mu_s \neq 0$. Also by the choice of t_0 and the induction hypothesis mentioned above we have $\sum_{s\in D_1\setminus C} \mu_s \operatorname{ev}^X_{D_1}(s) \in U$. Thus we have $\operatorname{ev}^X_D(t_0) \in U$. This concludes the proof. □

Remark 3.5. Let the notation be as in Lemma 3.4 (b) and ev^X_D surjective. There are the following interesting questions:

(Q_1) Under which conditions do we have $\operatorname{N}(I) \subseteq D$?

(Q_2) Under which conditions does $C = \operatorname{N}(I)$ satisfy (1), (2), and (3) in Lemma 3.4 (b)?

Of course, $C = \operatorname{N}(I)$ implies that $\operatorname{N}(I) \subseteq D$. A simple example that shows $\operatorname{N}(I) \subseteq D$ does not hold in general is given by $n = 1$, $X = \{1\} \subseteq K$, $D = \{\mathrm{x}_1\} \subseteq \operatorname{Mon}(S)$.

Definition 3.6. Let $<$ be a monomial order on $\operatorname{Mon}(S)$ and $D \subseteq \operatorname{Mon}(S)$ be an order ideal w.r.t. divisibility. We call D *distinguished* if for all $t \in D$ and $s \in \operatorname{Mon}(S) \setminus D$ we have $t < s$.

For an arbitrary non-empty order ideal $D \subseteq \operatorname{Mon}(S)$ we define

$$\partial(D) := (\mathrm{x}_1 D \cup \cdots \cup \mathrm{x}_n D) \setminus D.$$

We also set

$$\partial(\emptyset) := \{1\}.$$

Usually, $\partial(D)$ is called the *border of D*.

Example 3.7. Our standard example and a counterexample related to distinguished order ideals are the following.

(a) Let $d \in \mathbb{N}$. Then
$$D := \mathcal{T}_d = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \cdots + \alpha_n \leq d\}$$
is a distinguished order ideal w.r.t. $<_{\mathrm{degrevlex}}$ (or any other degree compatible monomial order).

(b) Clearly, for any $n \in \mathbb{N}$,
$$D := \mathcal{M}_d = \{\alpha \in \mathbb{N}^n \mid \max\{\alpha_1, \ldots, \alpha_n\} \leq d\}$$
is an order ideal. For $n \geq 2$ and $d \geq 1$, there is no monomial order $<$ on $\mathrm{Mon}(S)$ such that D is a distinguished order ideal w.r.t. $<$. Indeed, if $\mathrm{x}_2 > \mathrm{x}_1$, then $D \ni \mathrm{x}_2\mathrm{x}_1^d > \mathrm{x}_1\mathrm{x}_1^d = \mathrm{x}_1^{d+1} \notin D$.

It would be interesting to extend the results of this section to more general settings. Since this is outside the scope of this article, we omit this discussion here.

Lemma 3.8. *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ be finite and $D \subseteq \mathrm{Mon}(S)$ be a distinguished order ideal w.r.t. $<$ such that ev_D^X is surjective. Let $I := \mathrm{I}(X)$ and $C \subseteq D$ be as in Lemma 3.4 (b). For $t \in \mathrm{Mon}(S)$ let $p_t \in S_C$ be the uniquely determined polynomial such that $\mathrm{ev}_C^X(p_t) = \mathrm{ev}^X(t)$ and set $q_t := t - p_t$. Then the following holds:*

(a) *For $t \in \mathrm{Mon}(S)$ we have $q_t \in I$.*
(b) *For $t \in \mathrm{Mon}(S) \setminus C$ we have $\mathrm{supp}(p_t) \subseteq \{s \in C \mid s < t\}$.*
(c) *For $t \in \mathrm{Mon}(S) \setminus C$ we have $\mathrm{in}(q_t) = t$.*
(d) *For $p \in I \setminus \{0\}$ we have $\mathrm{supp}(p) \nsubseteq C$, i.e. $p \notin S_C$.*
(e) *We have $C = \mathrm{N}(I)$.*

Here, $\mathrm{supp}(p)$ denotes the support of p w.r.t. the monomial basis of S.

Proof. (a) This is an immediate consequence of the definition, since $I = \ker(\mathrm{ev}^X)$.

(b) If $t \in D \setminus C$ then there are $\mu_s \in K$ such that
$$\mathrm{ev}_D^X(t) = \sum_{s \in C, s<t} \mu_s \,\mathrm{ev}_C^X(s) = \mathrm{ev}_C^X\Big(\sum_{s \in C, s<t} \mu_s s\Big).$$
Hence $p_t = \sum_{s \in C, s<t} \mu_s s$, and clearly $\mathrm{supp}(p_t) \subseteq \{s \in C \mid s < t\}$. If $t \in \mathrm{Mon}(S) \setminus D$ then $t > s$ for all $s \in D$ since D is a distinguished order ideal. In particular, we see also in this case that $\mathrm{supp}(p_t) \subseteq C = \{s \in C \mid s < t\}$, finishing the proof of the claim.

(c) This is an immediate consequence of part (b).

(d) Suppose that $\mathrm{supp}(p) \subseteq C$. Then $p \in \mathrm{I}_C(X) = \ker(\mathrm{ev}_C^X) = \{0\}$, a contradiction.

(e) If $t \in \mathrm{Mon}(S) \setminus C$ then $t = \mathrm{in}(q_t) \in \mathrm{in}(I)$ by part (c). Thus $\mathrm{N}(I) \subseteq C$ and since $|\mathrm{N}(I)| = |X| = |C|$, we have $\mathrm{N}(I) = C$. □

Corollary 3.9. *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ be finite, $I := \mathrm{I}(X)$, and $D \subseteq \mathrm{Mon}(S)$ be a distinguished order ideal w.r.t. $<$. Then the following are equivalent:*

(i) *ev_D^X is surjective;*
(ii) *$\mathrm{N}(I) \subseteq D$.*

Proof. (i) $\Rightarrow$ (ii): Let $t \in \mathrm{N}(I)$ and let $C \subseteq D$ be as in Lemma 3.4 (b). Then we have $\mathrm{N}(I) = C$ by Lemma 3.8 (e) and thus $\mathrm{N}(I) \subseteq D$.

(ii) $\Rightarrow$ (i): By Lemma 3.4 (a), $\mathrm{ev}_{\mathrm{N}(I)}^X$ is bijective, and since $\mathrm{N}(I) \subseteq D$, ev_D^X is surjective. □

The special case of the next theorem for a degree compatible monomial order $<$ and $D = \mathcal{T}_d$ can already be found in [56, Theorem 2.48].

Theorem 3.10 (Möller)**.** *Let $<$ be a monomial order on $\mathrm{Mon}(S)$, $X \subseteq K^n$ finite, and D a distinguished order ideal w.r.t. $<$ such that ev_D^X is surjective. Then there is a Gröbner basis G of $\mathrm{I}(X)$ such that*

$$G \subseteq S_{D \cup \partial(D)} \text{ and } |G| = |D| + |\partial(D)| - |X|.$$

Proof. Let $I := \mathrm{I}(X)$ and let $C = \mathrm{N}(I) \subseteq D$, $p_t \in S_C$, and $q_t = t - p_t$ be as in Lemma 3.8.

Define

$$G := \{q_s \mid s \in D \cup \partial(D) \setminus C\} \subseteq I.$$

We show that G is a Gröbner basis of I. Set $J := \langle \mathrm{in}(G) \rangle_S$. It suffices to show that $J = \mathrm{in}(I)$. It is clear that $J \subseteq \mathrm{in}(I)$. The reverse inclusion is certainly true if $X = \emptyset$, since then

$$I = \langle 1 \rangle = \mathrm{in}(I),\ C = \emptyset,\ 1 \in D \cup \partial(D), \text{ and } 1 = \mathrm{in}(q_1) \in \mathrm{in}(G).$$

Thus let w.l.o.g. $X \neq \emptyset$. Assume that $\mathrm{in}(I) \not\subseteq J$. Then there is a monomial $s \in \mathrm{in}(I) \setminus J$. Let t be a minimal monomial generator of $\mathrm{in}(I)$ with $t \mid s$. Since $t \in \mathrm{in}(I)$ we have $t \notin \mathrm{N}(I) = C$.

Case 1: $t \in D$. Then $q_t \in G$ and $t = \mathrm{in}(q_t) \in \mathrm{in}(G)$, hence $s \in \langle \mathrm{in}(G) \rangle = J$, a contradiction.

Case 2: $t \notin D$. Since $X \neq \emptyset$ we have $t \neq 1$, so there is a $j \in \{1, \dots, n\}$ such that $\mathrm{x}_j \mid t$. Let $\widetilde{t} := t/\mathrm{x}_j$. Since t is a minimal generator of $\mathrm{in}(I)$, we have $\widetilde{t} \notin \mathrm{in}(I)$, so $\widetilde{t} \in C \subseteq D$. Hence, $t = \mathrm{x}_j \widetilde{t} \in (\mathrm{x}_j D) \setminus D \subseteq \partial(D) \subseteq \mathrm{in}(G)$. Thus we obtain that $s \in \langle \mathrm{in}(G) \rangle = J$, again a contradiction.

Thus we have $\mathrm{in}(I) \subseteq J$ and G is a Gröbner basis of I. By Lemma 3.8 it is clear that $|G| = |D \cup \partial(D) \setminus C| = |D| + |\partial(D)| - |X|$. Moreover, for $t \in D \cup \partial(D) \setminus C$ we have $\mathrm{supp}(q_t) = \{t\} \cup \mathrm{supp}(p_t) \subseteq \{t\} \cup \{s \in C \mid s < t\} \subseteq D \cup \partial(D)$, i.e., $q_t \in S_{D \cup \partial(D)}$, which concludes the proof. □

Note that in Theorem 3.10, in general G contains a border prebasis induced by $\partial(D)$. In particular, if the distinguished order ideal D equals $\mathrm{N}(I)$, then G is a border basis of I. See, e.g., Kreuzer-Robbiano [28, Section 6.4] for further details related to the theory of border bases.

We list two immediate consequences of Theorem 3.10 in the following corollary.

Corollary 3.11. *The following holds:*

(a) *With the notation and assumptions as in Theorem* 3.10*,* $\mathrm{I}_{D \cup \partial(D)}(X)$ *generates a radical ideal in S.*
(b) *If $\mathrm{ev}_{\mathcal{T}_d}^X$ is surjective then $\mathrm{I}_{\mathcal{T}_{d+1}}(X)$ generates a radical ideal in S.*

We have the following implications for Prony structures.

Corollary 3.12. *Given the setup of Definition* 1.3*, let $P(f)$ be a Prony structure for $f \in V$. Let $d \in \mathbb{N}$ be such that $\ker P_d(f) = \mathrm{I}_{\mathcal{J}_d}(\mathrm{supp}_u(f))$. If there is a distinguished order ideal D (w.r.t. some monomial order $<$ on* $\mathrm{Mon}(S)$*) such that $\mathrm{ev}_D^{\mathrm{supp}_u(f)}$ is surjective and $D \cup \partial(D) \subseteq \mathcal{J}_d$ then*

$$\langle \ker P_d(f) \rangle = \mathrm{I}(X).$$

In particular, $\langle \ker P_d(f) \rangle$ is a radical ideal in S.

4. Prony structures for multivariate exponential sums

In this section we discuss Prony structures for multivariate exponential sums based on Hankel-like and Toeplitz-like matrices. Because for the Toeplitz case we need evaluations also at negative arguments, we have to consider two different variants of exponentials. One has only non-negative arguments and no restrictions on the bases in K^n. The other one is defined also for negative (integer) arguments and the restriction that the bases lie on the algebraic torus $(K \setminus \{0\})^n$. Observe that it is not possible to define Toeplitz versions of Prony's method for the first variant.

That Prony's methods can be generalized to these settings was shown in Kunis-Peter-Römer-von der Ohe [31], Sauer [46], and Mourrain [36]. Here we provide a new perspective on these results. Prony structures are a common abstraction of both Hankel and Toeplitz variants of Prony's method.

The following notation generalizes the univariate case in Example 1.7. Here and in the following we write $\mathrm{e}_1, \dots, \mathrm{e}_n$ for the standard basis vectors of K^n.

Definition 4.1. Let K be a field and F a subfield of K.

(a) For $b \in K^n$, let

$$\exp_b \colon \mathbb{N}^n \to K, \quad \alpha \mapsto b^\alpha = \prod_{j=1}^{n} b_j^{\alpha_j},$$

denote the (n-*variate*) *exponential with base* b (*with domain* $\mathbb{N}^n$). For a subset $Y \subseteq K^n$ let $B_Y := \{\exp_b \mid b \in Y\}$. We denote the F-subvector space of $K^{\mathbb{N}^n}$ generated by B_Y with

$$\mathrm{Exp}^n_Y(F) := \langle B_Y \rangle_F.$$

We call the elements of $\mathrm{Exp}^n_Y(F)$ (n-*variate*) *exponential sums* (*with domain* $\mathbb{N}^n$). Furthermore, we denote by u_Y the function

$$u_Y \colon B_Y \to K^n, \quad \exp_b \mapsto (\exp_b(\mathrm{e}_1), \dots, \exp_b(\mathrm{e}_n)) = b.$$

Trivially, u_Y is injective.

(b) Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$. For $f \in \mathrm{Exp}^n_Y(F)$ and $d \in \mathbb{N}$ let

$$\mathrm{H}_d(f) := \mathrm{H}_{\mathcal{I},\mathcal{J},d}(f) := (f(\alpha+\beta))_{\substack{\alpha \in \mathcal{I}_d \\ \beta \in \mathcal{J}_d}} \in K^{\mathcal{I}_d \times \mathcal{J}_d}.$$

We will see in Theorem 4.4 that $\mathrm{H}_d(f)$ induces a Prony structure on the space of exponential sums $\mathrm{Exp}^n_Y(F)$, and that therefore the set B_Y is a basis of $\mathrm{Exp}^n_Y(F)$.

The following is a variation of Definition 4.1 where all bases b are restricted to lie on the algebraic torus $(K \setminus \{0\})^n$. This allows also for non-negative arguments, i.e. the exponentials are functions on the domain $\mathbb{Z}^n$. As a consequence it is possible to define not only sequences of Hankel-like but also of Toeplitz-like matrices associated to an exponential sum (with domain $\mathbb{Z}^n$). In order to avoid any possible confusion, we write out the definition in full.

Definition 4.2. Let K be a field and F a subfield of K.

(a) For $b \in (K \setminus \{0\})^n$, let

$$\exp_{\mathbb{Z},b} \colon \mathbb{Z}^n \to K, \quad \alpha \mapsto b^\alpha = \prod_{j=1}^{n} b_j^{\alpha_j},$$

denote the (n-*variate*) *exponential with base* b (*with domain* $\mathbb{Z}^n$). For a subset $Y \subseteq (K \setminus \{0\})^n$ let $B_{\mathbb{Z},Y} := \{\exp_{\mathbb{Z},b} \mid b \in Y\}$. We denote the F-subvector space of $K^{\mathbb{Z}^n}$ generated by $B_{\mathbb{Z},Y}$ with

$$\operatorname{Exp}^n_{\mathbb{Z},Y}(F) := \langle B_{\mathbb{Z},Y}\rangle_F.$$

We call the elements of $\operatorname{Exp}^n_{\mathbb{Z},Y}(F)$ (n-*variate*) *exponential sums* (*with domain* $\mathbb{Z}^n$). Furthermore, we denote by $u_{\mathbb{Z},Y}$ the function

$$u_{\mathbb{Z},Y}\colon B_{\mathbb{Z},Y} \to K^n, \quad \exp_{\mathbb{Z},b} \mapsto (\exp_{\mathbb{Z},b}(\mathrm{e}_1), \ldots, \exp_{\mathbb{Z},b}(\mathrm{e}_n)) = b.$$

Trivially, $u_{\mathbb{Z},Y}$ is injective.

(b) Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$. For $f \in \operatorname{Exp}^n_{\mathbb{Z},Y}(F)$ and $d \in \mathbb{N}$ let

$$\mathrm{T}_d(f) := \mathrm{T}_{\mathcal{I},\mathcal{J},d}(f) := (f(\beta-\alpha))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d\times\mathcal{J}_d}.$$

Since for $f \in \operatorname{Exp}^n_{\mathbb{Z},Y}(F)$ we clearly have $f|_{\mathbb{N}^n} \in \operatorname{Exp}^n_Y(F)$, we also set

$$\mathrm{H}_d(f) := \mathrm{H}_{\mathcal{I},\mathcal{J},d}(f) := \mathrm{H}_{\mathcal{I},\mathcal{J},d}(f|_{\mathbb{N}^n}).$$

Lemma 4.3. *Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$. Then the following holds:*

(a) *Let $Y \subseteq K^n$ and $u = u_Y$. For $f \in \operatorname{Exp}^n_Y(F)$, $f = \sum_{b\in M} f_b b$ with $M \subseteq B_Y$ finite and $f_b \in F$, we have*

$$\mathrm{H}_d(f) = \left(\mathrm{V}^{u(M)}_{\mathcal{I}_d}\right)^{\top} \cdot C_f \cdot \mathrm{V}^{u(M)}_{\mathcal{J}_d}.$$

Here $\mathrm{V}^{u(M)}_{\mathcal{I}_d} \in K^{u(M)\times\mathcal{I}_d}$ denotes the matrix of $\operatorname{ev}^{u(M)}_{\mathcal{I}_d}$ w.r.t. the monomial basis of $S_{\mathcal{I}_d}$ and the canonical basis of $K^{u(M)}$. The matrix $C_f \in F^{u(M)\times u(M)}$ is the diagonal matrix with the non-zero coefficients f_b of f on the "diagonal".

(b) *Let $Y \subseteq (K\setminus\{0\})^n$ and $u = u_{\mathbb{Z},Y}$. For $f \in \operatorname{Exp}^n_{\mathbb{Z},Y}(F)$, $f = \sum_{b\in M} f_b b$ with $M \subseteq B_{\mathbb{Z},Y}$ finite and $f_b \in F$, we have*

$$\mathrm{T}_d(f) = \left(\mathrm{V}^{1/u(M)}_{\mathcal{I}_d}\right)^{\top} \cdot C_f \cdot \mathrm{V}^{u(M)}_{\mathcal{J}_d}.$$

Here $\mathrm{V}^{1/u(M)}_{\mathcal{I}_d} \in K^{1/u(M)\times\mathcal{I}_d}$ denotes the matrix of $\operatorname{ev}^{\{1/b \mid b\in u(M)\}}_{\mathcal{I}_d}$ and C_f and $\mathrm{V}^{u(M)}_{\mathcal{J}_d}$ are as in part (a)*.*

Proof. This follows by straightforward computations; see, e.g., [56, Lemma 2.7 (a)] for part (a) and [56, Lemma 2.32 (a)] for part (b), respectively. □

The following theorem is a multivariate variant of Prony's method (cf. Example 1.7).

Theorem 4.4 (Prony structures for exponential sums)**.** *Let K be a field. Let $\mathcal{J}$ be a sequence of finite subsets of $\mathbb{N}^n$ such that $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ for all large d and $\bigcup_{d\in\mathbb{N}} \mathcal{J}_d = \mathbb{N}^n$. Let the sequence $\mathcal{I}$ be defined by $\mathcal{I}_d := \mathcal{J}_{\ell(d)}$ for an unbounded monotonous sequence $\ell\colon \mathbb{N}\to\mathbb{N}$. Then the following hold, with $Y \subseteq K^n$ in* (a) *and $Y \subseteq (K\setminus\{0\})^n$ in* (b) *and* (c)*:*

(a) *The map $f \mapsto (\mathrm{H}_d(f))_{d\in\mathbb{N}}$ induces a Prony structure on $\operatorname{Exp}^n_Y(F)$.*
(b) *The map $f \mapsto (\mathrm{T}_d(f))_{d\in\mathbb{N}}$ induces a Prony structure on $\operatorname{Exp}^n_{\mathbb{Z},Y}(F)$.*
(c) *The map $f \mapsto (\mathrm{H}_d(f))_{d\in\mathbb{N}}$ induces a Prony structure on $\operatorname{Exp}^n_{\mathbb{Z},Y}(F)$.*

Proof. In every case we write $u = u_Y$ and $u = u_{\mathbb{Z},Y}$, respectively.

(a) Let $f \in \operatorname{Exp}^n_Y(F)$, $M \subseteq B_Y$ finite, and $(f_b)_{b\in M} \in (F\setminus\{0\})^M$ such that $f = \sum_{b\in M} f_b b$. We will verify that condition (ii) of Theorem 2.4 holds for f and M as described in Remark 2.5. In particular, it then follows that B_Y is an F-basis of $\operatorname{Exp}^n_Y(F)$.

By the assumptions on $\mathcal{J}$ and $\mathcal{I}$ and Lemma 2.3, $\operatorname{ev}^M_{\mathcal{I}_d}$ is surjective for all large d and thus $(\operatorname{ev}^M_{\mathcal{I}_d})^{\top}$ is injective.

Hence, by Lemma 4.3 (a), for all large d we have the following commutative diagram.

$$\begin{array}{ccccccc} K^{\mathcal{J}_d} & & & \xrightarrow{\quad \mathrm{H}_d(f) \quad} & & & K^{\mathcal{I}_d} \\ \downarrow \cong & & & & & & \uparrow \cong \\ S_{\mathcal{J}_d} & \xrightarrow{\mathrm{ev}^M_{\mathcal{J}_d}} & K^M & \xrightarrow{C_f,\, \cong} & K^M & \xhookrightarrow{(\mathrm{ev}^M_{\mathcal{I}_d})^\top} & S_{\mathcal{I}_d} \end{array}$$

Thus, the assertion follows immediately from Theorem 2.4 together with Remark 2.5.

(b) This follows analogously to part (a) using the elementary fact that

$$\operatorname{rank}\Big(\mathrm{V}^{1/M}_{\mathcal{I}_d}\Big) = \operatorname{rank}\Big(\mathrm{V}^{M}_{\mathcal{I}_d}\Big)$$

(cf. [56, Lemma 2.31]) and with Lemma 4.3 (a) replaced by Lemma 4.3 (b).

(c) This follows immediately from part (a). ☐

In particular, for $Y \subseteq K^n$ and $f \in \mathrm{Exp}^n_Y(F)$ the notation $\operatorname{supp}_{u_Y}(f)$ is justified by Theorem 4.4 (and analogously for $Y \subseteq (K \setminus \{0\})^n$ and $f \in \mathrm{Exp}^n_{\mathbb{Z},Y}(F)$).

Remark 4.5. As mentioned above, one advantage of the Hankel Prony structure H over the Toeplitz Prony structure T is that H works with exponential sums with arbitrary bases in K^n while T needs bases in $(K \setminus \{0\})^n$.

On the other hand, some relevant results in this context are known only for Toeplitz matrices; see, e.g., [31, Theorem 3.7].

In the spirit of Díaz-Kaltofen [13] and Garg-Schost [19], we discuss one additional advantage of the Toeplitz variant regarding the number of used evaluations. Let K be a field extension of F. Let I be a set, $V \leq K^I$ be an F-vector space of functions $I \to K$ and B be a basis of V. Moreover, let $\varphi\colon K \to K$ be an F-automorphism of K such that for $b \in B$ we have $\varphi \circ b \in B$. Further, assume that a subset $I_0 \subseteq I$ is given together with a function $\psi\colon I \to I$ such that $\psi(I_0) \subseteq I_1 := I \setminus I_0$ and for every $f \in V$ the following diagram is commutative:

$$\begin{array}{ccc} I & \xrightarrow{\quad f \quad} & K \\ \downarrow \psi & & \uparrow \varphi \\ I & \xrightarrow{\quad f \quad} & K \end{array}$$

(It is of course sufficient to check this diagram for every $f = b \in B$.) Thus, under these assumptions, one can replace the evaluations of f at $\alpha \in I_0$ by evaluations of φ at $f(\psi(\alpha))$. One does not need to evaluate at any element of I_0.

An application is the case $F = \mathbb{R}$, $K = \mathbb{C}$, and the space $V = \mathrm{Exp}^n_{\mathbb{Z},\mathbb{T}^n}(\mathbb{R})$ of exponential sums with real coefficients supported on the analytic torus

$$\mathbb{T}^n = \{z \in \mathbb{C}^n \mid |z_j| = 1 \text{ for } j = 1, \ldots, n\} \subseteq \mathbb{C}^n.$$

Take $\varphi\colon \mathbb{C} \to \mathbb{C}$ to be the complex conjugation and let $I = \mathbb{Z}^n$, $\psi\colon I \to I$, $\alpha \mapsto -\alpha$, with $I_0 = \{\alpha \in I \mid \alpha_1 < 0\}$. In this case, one can often define the Toeplitz matrix $\mathrm{T}_{\mathcal{I},\mathcal{J},d}(f)$ using fewer evaluations than in the Hankel matrix $\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f)$.

Let $f \in \mathrm{Exp}^n_{\mathbb{Z},(K\setminus\{0\})^n}(F)$ be arbitrary. Then the number $s_{\mathrm{H},\mathcal{I},\mathcal{J},d}$ of evaluations needed to define the Hankel matrix $\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f)$ can be different from the number $s_{\mathrm{T},\mathcal{I},\mathcal{J},d}$ of evaluations needed to define $\mathrm{T}_{\mathcal{I},\mathcal{J},d}(f)$, depending on the choice of $\mathcal{I}$ and $\mathcal{J}$. In general one has

$$s_{\mathrm{H},\mathcal{I},\mathcal{J},d} = |\mathcal{I}_d + \mathcal{J}_d| \quad \text{and} \quad s_{\mathrm{T},\mathcal{I},\mathcal{J},d} = |\mathcal{J}_d - \mathcal{I}_d|.$$

Thus for example, in the bivariate case $n = 2$ one has

$$s_{\mathrm{H},\mathcal{M},\mathcal{M},d} = s_{\mathrm{T},\mathcal{M},\mathcal{M},d} \text{ for all } d$$

and

$$s_{\mathrm{H},\mathcal{T},\mathcal{T},2} = 15 \neq 19 = s_{\mathrm{T},\mathcal{T},\mathcal{T},2}.$$

A more detailed discussion of this fact can be found in Josz-Lasserre-Mourrain [25, Section 2.3.2].

It would be interesting to compare Prony indices $\operatorname{ind}_{\mathrm{H}_{\mathcal{I},\mathcal{J}}}(f)$ and $\operatorname{ind}_{\mathrm{T}_{\mathcal{I},\mathcal{J}}}(f)$ of $f \in \operatorname{Exp}^n_{\mathbb{Z},Y}(F)$ for various choices of the involved parameters.

5. Applications of Prony structures

In this section we discuss several reconstruction techniques in the context of Prony structures, namely the Dress-Grabmeier framework [16], the Grigoriev-Karpinski-Singer [20] and the related Peter-Plonka framework [37] (see also Remark 1.5), sparse polynomial interpolation w.r.t. the monomial (Ben-Or/Tiwari [4]) and Chebyshev bases [32, 44, 24, 23] and a sparse technique for Gaußian sums [38].

The following theorem casts the Dress-Grabmeier framework [16] for sparse interpolation of character sums in terms of Prony structures.

Theorem 5.1 (Prony structure for character sums). *Let $(M,+)$ be a commutative monoid generated by elements $a_1,\dots,a_n \in M$. Consider a set B of monoid homomorphisms (i.e.,* characters*) from M to $(K,\cdot)$, and let $V := \langle B\rangle$ be the K-subvector space of K^M generated by B. Let*

$$u\colon B \to K^n, \quad \chi \mapsto (\chi(a_1),\dots,\chi(a_n)).$$

Let $\mathcal{I},\mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$ with $\mathcal{I}_d \subseteq \mathcal{I}_{d+1}$ and $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ for all large d and $\bigcup_{d\in\mathbb{N}}\mathcal{I}_d = \bigcup_{d\in\mathbb{N}}\mathcal{J}_d = \mathbb{N}^n$. For $f \in V$ and $d \in \mathbb{N}$ set

$$P_d(f) := \left(f\Big(\sum_{j=1}^{n}(\alpha_j+\beta_j)a_j\Big)\right)_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d\times\mathcal{J}_d}.$$

Then $P_d(f)$ induces a Prony structure on V.

Proof. If $u(\chi_1) = u(\chi_2)$ for characters χ_i then $\chi_1(a_j) = \chi_2(a_j)$ for all $j = 1,\dots,n$. Since M is generated by $\{a_1,\dots,a_n\}$ this implies $\chi_1 = \chi_2$, and thus u is injective. For $f \in V$ write $f = \sum_{x\in\operatorname{supp}_u(f)} f_x\chi_x$ with $f_x \in K$ and $\chi_x \in B$ with $u(\chi_x) = x$. Let $C := (f_x \mathrm{e}_x)_{x\in\operatorname{supp}_u(f)} \in K^{\operatorname{supp}_u(f)\times\operatorname{supp}_u(f)}$. A computation on the corresponding matrices shows that one has the following commutative diagram:

$$\begin{array}{ccccccc}
K^{\mathcal{J}_d} & & & \xrightarrow{\quad P_d(f)\quad} & & & K^{\mathcal{I}_d}\\
\downarrow\cong & & & & & & \uparrow\cong\\
S_{\mathcal{J}_d} & \xrightarrow{\operatorname{ev}^{\operatorname{supp}_u(f)}_{\mathcal{J}_d}} & K^{\operatorname{supp}_u(f)} & \xrightarrow{C,\,\cong} & K^{\operatorname{supp}_u(f)} & \xrightarrow{(\operatorname{ev}^{\operatorname{supp}_u(f)}_{\mathcal{I}_d})^{\top}} & S_{\mathcal{I}_d}
\end{array}$$

Clearly, C is invertible, and thus P is a Prony structure on V by Lemma 2.3, Theorem 2.4, and Remark 2.5. ☐

Remark 5.2.
(a) Since $\exp_b \in \operatorname{Hom}((\mathbb{N}^n,+),(K,\cdot))$, the Dress-Grabmeier framework contains the Prony structures for exponential sums.
(b) Note that Dress-Grabmeier allows more generally arbitrary monoids whereas in Theorem 5.1 we allow only finitely generated ones. Roughly speaking, in applications to function spaces this corresponds to allowing only a fixed finite number n of variables. This is no restriction in any case we have in mind.
(c) Note that Dress-Grabmeier implies the Dedekind independence lemma, i.e., that any set of monoid characters is linearly independent.

Next we present a family of methods that was given in the case of one operator in Peter-Plonka [37]. See also Mourrain [36] for related discussions in the multivariate case and the book of Plonka, Potts, Steidl, and Tasche [40, Section 10.4.2]. Essentially, it is a generalization of the framework given by Grigoriev, Karpinski, and Singer [20] for the case of Δ being a point evaluation functional. We derive our statement directly from Theorem 5.1.

As usual, the point spectrum of an endomorphism $\varphi \in \operatorname{End}_K(W)$ of a K-vector space W is denoted by

$$\sigma_{\mathrm{p}}(\varphi) = \{\lambda \in K \mid \ker(\varphi - \lambda \operatorname{id}_W) \neq \{0\}\}$$

and for $\lambda \in \sigma_{\mathrm{p}}(\varphi)$ let

$$W_\lambda^\varphi = \ker(\varphi - \lambda \operatorname{id}_W)$$

be the eigenspace of φ w.r.t. λ. For pairwise commuting operators $\varphi_1, \ldots, \varphi_n \in \operatorname{End}_K(W)$ and $\alpha \in \mathbb{N}^n$ we use the notation

$$\varphi^\alpha := \varphi_1^{\alpha_1} \circ \cdots \circ \varphi_n^{\alpha_n} \in \operatorname{End}_K(W).$$

Corollary 5.3 (Prony structure for eigenvector sums). *Let $\varphi_1, \ldots, \varphi_n \in \operatorname{End}_K(W)$ be pairwise commuting operators and consider $\Lambda \subseteq \prod_{j=1}^n \sigma_{\mathrm{p}}(\varphi_j)$. Assume that for every $\lambda \in \Lambda$ we have $\bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \neq \{0\}$ and choose*

$$b_\lambda \in \bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \setminus \{0\}.$$

Let

$$B := \{b_\lambda \mid \lambda \in \Lambda\}, \quad V := \langle B \rangle_K, \quad \text{and} \quad u\colon B \to K^n, \quad b_\lambda \mapsto \lambda.$$

Let $\Delta \in W^* = \operatorname{Hom}_K(W, K)$ be such that

$$V \cap \ker(\Delta) = \{0\}.$$

Let $\mathcal{I}, \mathcal{J}$ be sequences of finite subsets of $\mathbb{N}^n$ with $\mathcal{I}_d \subseteq \mathcal{I}_{d+1}$ and $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ for all large d and $\bigcup_{d\in\mathbb{N}} \mathcal{I}_d = \bigcup_{d\in\mathbb{N}} \mathcal{J}_d = \mathbb{N}^n$. For $f \in V$ and $d \in \mathbb{N}$ set

$$P_d(f) := (\Delta(\varphi^{\alpha+\beta}(f)))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d\times\mathcal{J}_d}.$$

Then $P_d(f)$ induces a Prony structure on V.

Proof. We apply Theorem 5.1 similarly as in Grigoriev, Karpinski, and Singer [20, p. 78f]. Let M denote the submonoid of $(\operatorname{End}_K(W), \circ)$ generated by $\varphi_1, \ldots, \varphi_n$. For $\lambda \in \Lambda$ let

$$\chi_\lambda\colon M \to K, \quad \varphi^\alpha \mapsto \frac{\Delta(\varphi^\alpha(b_\lambda))}{\Delta(b_\lambda)}.$$

Clearly, χ_λ is well-defined. Since $\chi_\lambda(\varphi^\alpha) = \lambda^\alpha$ for every $\alpha \in \mathbb{N}^n$, χ_λ is a monoid homomorphism $M \to (K, \cdot)$. Thus, by Theorem 5.1, $Q_d(f) = (f(\varphi^{\alpha+\beta}))_{\alpha\in\mathcal{I}_d,\beta\in\mathcal{J}_d}$ induces a Prony structure on the vector space $U := \langle \chi_\lambda \mid \lambda \in \Lambda \rangle_K \leq K^M$ with respect to $v\colon \chi_\lambda \mapsto \lambda$. Since $\operatorname{supp}_u(b_\lambda) = \lambda = \operatorname{supp}_v(\chi_\lambda)$, the assertion follows. □

Observe that there are interesting situations where the condition that the b_λ's can be chosen in the desired way is fulfilled. For example this is the case if W is finite-dimensional $\mathbb{C}$-vector space see, e.g., Horn-Johnson [22, Lemma 1.3.19].

With a little more effort in a direct proof, one can avoid the commutativity assumption in Corollary 5.3 (but of course one still needs that $\bigcap_{j=1}^n W_{\lambda_j}^{\varphi_j} \neq \{0\}$ for every $\lambda \in \Lambda$).

The case $n = 1$ identifies the method in [37] as a Prony structure.

Corollary 5.4 (Peter-Plonka [37, Theorem 2.1]). *Let* $\varphi \in \operatorname{End}_K(W)$ *and consider* $\Lambda \subseteq \sigma_{\mathrm{p}}(\varphi)$. *For* $\lambda \in \Lambda$ *choose*

$$b_\lambda \in W_\lambda^\varphi \setminus \{0\}.$$

Let

$$B := \{b_\lambda \mid \lambda \in \Lambda\}, \quad V := \langle B \rangle_K, \quad \text{and} \quad u\colon B \to K, \quad b_\lambda \mapsto \lambda.$$

Let $\Delta \in W^*$ *be such that*

$$V \cap \ker(\Delta) = \{0\}.$$

For $f \in V$ *and* $d \in \mathbb{N}$ *set*

$$P_d(f) := \left(\Delta(\varphi^{\alpha+\beta}(f))\right)_{\substack{\alpha=0,\dots,d-1\\ \beta=0,\dots,d}} \in K^{d\times(d+1)}.$$

Then $P_d(f)$ *induces a Prony structure on* V.

Proof. Take $n = 1$, $\mathcal{I}_d = \mathcal{T}_{d-1}$ and $\mathcal{J}_d = \mathcal{T}_d$ in Corollary 5.3. □

Example 5.5. Several applications for various choices of the endomorphism φ and the functional Δ can be found in [37], for example, with $\varphi \in \operatorname{End}(W)$ chosen as a Sturm-Liouville differential operator ($W = \mathrm{C}^\infty(\mathbb{R})$) or as a diagonal matrix with distinct elements on the diagonal ($W = K^n$).

Remark 5.6. Besides Corollary 5.4, Peter-Plonka [37, Theorem 2.4] extended their method, e.g., to include generalized eigenvectors and multiplicities; see also Mourrain [36] and Stampfer-Plonka [52]. At present Prony structures do not cover this variation. Since all examples we have in mind and which are discussed in this manuscript do not use generalized eigenvectors and multiplicities, we omit a detailed discussion here. See also Remark 1.9.

The following lemma singles out a simple transfer principle for Prony structures that will be applied in Corollary 5.8 and Corollary 5.13. It is also one motivation for the introduction of Prony maps in Section 7.

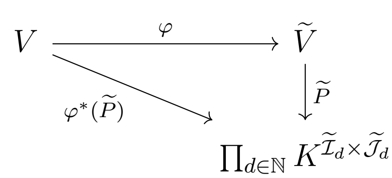
Lemma 5.7 (Transfer principle for Prony structures). *Let* $V, \widetilde{V}$ *be* F*-vector spaces with bases* $B, \widetilde{B}$*, respectively, and let* $u\colon B \to K^n$ *and* $\widetilde{u}\colon \widetilde{B} \to K^n$ *be injective. Let* $\varphi\colon V \to \widetilde{V}$ (*not necessarily linear*) *and for every* $f \in V$ *let*

$$\operatorname{supp}_u(f) = \operatorname{supp}_{\widetilde{u}}(\varphi(f)).$$

Then every Prony structure $\widetilde{P}$ *on* $\widetilde{V}$ *induces a Prony structure* $\varphi^*(\widetilde{P})$ *on* V *with*

$$\varphi^*(\widetilde{P})_d(f) := \widetilde{P}_d(\varphi(f))$$

for $f \in V$ *and* $d \in \mathbb{N}$. *The following commutative diagram illustrates the situation.*



Proof. Let $P := \varphi^*(\widetilde{P})$. By the hypotheses, for $f \in V$ and all large d we have

$$\operatorname{supp}_u(f) = \operatorname{supp}_{\widetilde{u}}(\varphi(f)) = \mathrm{Z}(\ker \widetilde{P}_d(\varphi(f))) = \mathrm{Z}(\ker P_d(f))$$

and

$$\mathrm{I}_{\widetilde{\mathcal{J}}_d}(\operatorname{supp}_u(f)) = \mathrm{I}_{\widetilde{\mathcal{J}}_d}(\operatorname{supp}_{\widetilde{u}}(\varphi(f))) \subseteq \ker(\widetilde{P}_d(\varphi(f))) = \ker(P_d(f)).$$

This concludes the proof. □

The following corollary identifies a well-known sparse interpolation technique for polynomials w.r.t. the monomial basis (see, e.g., [36, Section 5.4]) as a Prony structure. In particular, the framework of Prony structures allows a simultaneous proof of the Hankel and Toeplitz cases. There are analogous results for the Chebyshev basis (see Corollary 5.13).

Let F be a field and consider

$$V := F[\mathrm{y}_1, \ldots, \mathrm{y}_n]$$

as an F-vector space with the monomial basis

$$B := \{\mathrm{y}^\alpha \mid \alpha \in \mathbb{N}^n\}.$$

Choose a field extension K of F and let $b \in (K \setminus \{0\})^n$ be such that the function

$$u\colon B \to K^n, \quad \mathrm{y}^\alpha \mapsto (b_1^{\alpha_1}, \ldots, b_n^{\alpha_n}),$$

is injective.[2] Observe that then necessarily $u(B) \subseteq (K \setminus \{0\})^n$.

Moreover, set $\widetilde{V} := \operatorname{Exp}^n_{\mathbb{Z},u(B)}(F)$, $\widetilde{B} := \{\exp_b \mid b \in u(B)\}$, and $\widetilde{u}\colon \widetilde{B} \to K^n$, $\exp_b \mapsto b$.

Corollary 5.8 (Prony structures for sparse polynomial interpolation)**.** *For $p \in V$ let*

$$f_p\colon \mathbb{Z}^n \to K, \quad \alpha \mapsto p(b_1^{\alpha_1}, \ldots, b_n^{\alpha_n}) \ (= p(u(\mathrm{y}^\alpha)) \text{ if } \alpha \in \mathbb{N}^n).$$

Then the following holds:

(a) *For all $p \in V$ we have $f_p \in \widetilde{V}$ and $\varphi\colon V \to \widetilde{V}$, $p \mapsto f_p$, is F-linear.*
(b) *For all $p \in V$ we have $\operatorname{supp}_{\widetilde{u}}(f_p) = \operatorname{supp}_u(p)$.*

Hence, any Prony structure on $\widetilde{V}$ (in particular the Prony structures from Theorem 4.4), induces a Prony structure on V by the transfer principle (Lemma 5.7).

Proof. (a) Let $\operatorname{supp}(p) = \{\beta \in \mathbb{N}^n \mid \mathrm{y}^\beta \in \operatorname{supp}_B(p)\}$. For $\alpha \in \mathbb{Z}^n$ and using Definition 4.1 we have

$$f_p(\alpha) = \sum_{\beta \in \operatorname{supp}(p)} p_\beta \cdot (b_1^{\alpha_1}, \ldots, b_n^{\alpha_n})^\beta = \sum_{\beta \in \operatorname{supp}(p)} p_\beta \cdot (b_1^{\beta_1}, \ldots, b_n^{\beta_n})^\alpha = \sum_{\beta \in \operatorname{supp}(p)} p_\beta \cdot \exp_{\mathbb{Z},u(\mathrm{y}^\beta)}(\alpha).$$

This shows that $f_p \in \operatorname{Exp}^n_{\mathbb{Z},u(B)}(F) = \widetilde{V}$. In particular, φ is well-defined. The linearity of φ follows immediately from the definition.

(b) Since u is injective, the computation in the proof of part (a) shows that

$$\operatorname{supp}_{\widetilde{u}}(f_p) = \{u(\mathrm{y}^\beta) \mid \beta \in \operatorname{supp}(p)\} = \{u(m) \mid m \in \operatorname{supp}_B(p)\} = \operatorname{supp}_u(p).$$

This concludes the proof. □

Example 5.9. The reconstruction method for $p \in V = F[\mathrm{y}_1, \ldots, \mathrm{y}_n]$ from Corollary 5.8 is efficient if p has small rank, i.e., is a "sparse polynomial". To give an illustration, let $n = 2$, $b \in (K \setminus \{0\})^n$ be chosen appropriately and $p = \mathrm{y}^\beta - \mathrm{y}^\gamma \in V$ be a binomial. Then $\operatorname{rank}(f_p) = 2$, hence the polynomial p can be reconstructed, independently of its degree, from the $|\mathcal{T}_3| = \binom{n+3}{3} = \binom{5}{3} = 10$ evaluations used for the matrix $\mathrm{H}_{\mathcal{T}_1,\mathcal{T}_2}(f_p)$.

The number of evaluations of p can be further reduced if p is known to be of degree at most $d - 1$. In this case, $q := p(\mathrm{z}, \mathrm{z}^d, \ldots, \mathrm{z}^{d^{n-1}}) \in F[\mathrm{z}]$ is a binomial of degree at most $d^n - 1$ in *one* variable. The above binomial can thus be reconstructed from four evaluations.

[2]For example, for $K = F = \mathbb{C}$, any $b \in \mathbb{C}^n$ such that $b_j \neq 0$ and b_j is not a root of unity for all $j = 1, \ldots, n$ works. Of course, K cannot be finite, for otherwise $u\colon B \to K^n$ cannot be injective. One may always choose $K := F(\mathrm{w})$ (with w an indeterminate over F) and $b := (\mathrm{w}, \ldots, \mathrm{w}) \in K^n$.

Let $\mathrm{T}_i \in \mathbb{Z}[\mathrm{y}]$ denote the i-th Chebyshev polynomial (i.e., $\mathrm{T}_0 = 1$, $\mathrm{T}_1 = \mathrm{y}$, and $\mathrm{T}_i = 2\mathrm{y}\mathrm{T}_{i-1} - \mathrm{T}_{i-2}$ for $i \geq 2$). It is well-known (and immediate) that $B := \{\mathrm{T}_i \mid i \in \mathbb{N}\}$ is a $\mathbb{Q}$-basis of $V := \mathbb{Q}[\mathrm{y}]$.

Decomposing a polynomial $f \in \mathbb{Q}[\mathrm{y}]$ w.r.t. the Chebyshev basis B is in principle possible by first decomposing f in terms of the monomial basis (Corollary 5.8) and then computing the Chebyshev decomposition from that. However, the natural assumption of an upper bound on the rank of f w.r.t. B does not imply an upper bound on the rank of f w.r.t. the monomial basis, so that it may be impossible to check the premises of Corollary 5.8. Even if such a bound were given, efficiency would be a concern. Lakshman and Saunders [32] proposed a sparse method to compute Chebyshev decompositions directly, which we recast in the framework of Prony structure in the following. We first prove a Prony structure for an analogue of exponential sums in the Chebyshev setting (Theorem 5.12). The Prony structure for Chebyshev-sparse polynomial interpolation of Lakshman and Saunders [32] then follows in exactly the same way as for "monomial-sparse" polynomial interpolation (Corollary 5.13).

As observed in Lakshman-Saunders [32, p. 390], the crucial properties of the Chebyshev polynomials for their Prony structures are that for all $i, j \in \mathbb{N}$ one has the *linearization relation*

$$\mathrm{T}_i \cdot \mathrm{T}_j = \frac{1}{2}(\mathrm{T}_{i+j} + \mathrm{T}_{|i-j|}) \tag{2}$$

and the *commutativity relation*

$$\mathrm{T}_i(\mathrm{T}_j) = \mathrm{T}_j(\mathrm{T}_i). \tag{3}$$

The following definition is the Chebyshev analogue of the exponentials of Section 4.

Definition 5.10. Let F be a field of characteristic zero and K be a field extension of F. For $b \in K$ call the function

$$\mathrm{txp}_b \colon \mathbb{N} \to K, \quad i \mapsto \mathrm{T}_i(b),$$

Chebyshev exponential with base b and for a subset $Y \subseteq K$ denote by

$$\mathrm{Txp}_Y(F) := \langle \mathrm{txp}_b \mid b \in Y \rangle_F$$

the F-vector space of *Chebyshev exponential sums with bases in* Y.

Remark 5.11. Observe that considered merely as vector spaces, $\mathrm{Exp}_Y(F)$ and $\mathrm{Txp}_Y(F)$ are identical. However, here we consider them equipped with the bases of exponentials and Chebyshev exponentials, respectively, and provide the notation to keep track of this difference.

Theorem 5.12 (Prony structures for Chebyshev exponential sums)**.** *For* $f \in \mathrm{Txp}_Y(F)$ *and* $d \in \mathbb{N}$ *let*

$$P'_d(f) := (f(i+j) + f(|i-j|))_{\substack{i=0,\dots,d-1\\ j=0,\dots,d}} \in K^{d\times(d+1)}$$

(*which is the sum of a Hankel and a Toeplitz matrix*)*. Let* $\psi \in \mathbb{Q}^{(d+1)\times(d+1)}$ *be the change of basis from the monomial to the Chebyshev basis and*

$$P_d(f) := P'_d(f) \cdot \psi.$$

Then $P_d(f)$ *induces a Prony structure on* $\mathrm{Txp}_Y(F)$ *w.r.t.*

$$u \colon B \to K, \quad \mathrm{txp}_b \mapsto \mathrm{txp}_b(1) = b.$$

Proof. The injectivity of u follows immediately from the definition.

Let $S := K[\mathrm{x}]$. The lower part of the following diagram is commutative by a computation analogous to Lakshman-Saunders [32, proof of Lemma 6] (using the linearization relation (2) above), where the vertical isomorphisms are those given by the basis $\{\mathrm{T}_0, \dots, \mathrm{T}_d\}$ of $S_{\leq d}$ and C is the isomorphism given by the diagonal matrix $C := (2f_T \mathrm{e}_T)_{T \in \operatorname{supp}_B(f)}$.

$$\begin{array}{ccccccc}
K^{d+1} & & & \xrightarrow{\quad P_d(f) \quad} & & & K^d \\
\big\downarrow{\scriptstyle \psi,\, \cong} & & & & & & \big\Vert \\
K^{d+1} & & & \xrightarrow{\quad P'_d(f) \quad} & & & K^d \\
\big\downarrow{\scriptstyle \cong} & & & & & & \big\uparrow{\scriptstyle \cong} \\
S_{\leq d} & \xrightarrow{\mathrm{ev}_{\leq d}^{\operatorname{supp}_u(f)}} & K^{\operatorname{supp}_u(f)} & \xrightarrow{C,\, \cong} & K^{\operatorname{supp}_u(f)} & \xrightarrow{(\mathrm{ev}_{\leq d-1}^{\operatorname{supp}_u(f)})^\top} & S_{\leq d-1}
\end{array}$$

The upper part of the diagram is commutative by the definition of $P_d(f)$ and thus the assertion follows from Lemma 2.3, Theorem 2.4, and Remark 2.5. ☐

It is now straightforward to derive a well-known sparse interpolation technique for polynomials w.r.t. the Chebyshev basis (see, e.g., Lakshman-Saunders [32]) by transferring the Prony structure for Chebyshev exponential sums from Theorem 5.12 to the space of polynomials using Lemma 5.7. To this end, let F be a field of characteristic zero and consider

$$V := F[\mathrm{y}]$$

as an F-vector space with the Chebyshev basis

$$B := \{\mathrm{T}_i \mid i \in \mathbb{N}\}.$$

Choose a field extension K of F and let $b \in K$ be such that the function

$$u\colon B \to K, \quad \mathrm{T}_i \mapsto \mathrm{T}_i(b),$$

is injective.[3]

Moreover, set $\widetilde{V} := \mathrm{Txp}_{u(B)}(F)$, $\widetilde{B} := \{\mathrm{txp}_b \mid b \in u(B)\}$, and $\widetilde{u}\colon \widetilde{B} \to K$, $\mathrm{txp}_b \mapsto b$.

Corollary 5.13 (Prony structure for Chebyshev-sparse polynomial interpolation)**.** *For $p \in V$ let*

$$f_p\colon \mathbb{N} \to K, \quad i \mapsto p(u(\mathrm{T}_i)).$$

Then the following holds:

(a) *For all $p \in V$ we have $f_p \in \widetilde{V}$ and $\varphi\colon V \to \widetilde{V}$, $p \mapsto f_p$, is F-linear.*
(b) *For all $p \in V$ we have $\operatorname{supp}_{\widetilde{u}}(f_p) = \operatorname{supp}_u(p)$.*

Hence, any Prony structure on $\widetilde{V}$ (in particular the Prony structure from Theorem 5.12), induces a Prony structure on V by the transfer principle (Lemma 5.7).

Proof. (a) Let $\operatorname{supp}(p) = \{j \in \mathbb{N} \mid \mathrm{T}_j \in \operatorname{supp}_B(p)\}$. Using the commutativity relation (3) mentioned above, for $i \in \mathbb{N}$ we have

$$f_p(i) = \sum_{j \in \operatorname{supp}(p)} p_j \cdot \mathrm{T}_j(\mathrm{T}_i(b)) = \sum_{j \in \operatorname{supp}(p)} p_j \cdot \mathrm{T}_i(\mathrm{T}_j(b)) = \sum_{j \in \operatorname{supp}(p)} p_j \cdot \mathrm{txp}_{u(\mathrm{T}_j)}(i).$$

This shows that $f_p \in \widetilde{V}$. In particular, φ is well-defined. The linearity of φ follows immediately from the definition.

[3] A choice that always works is $b \in \mathbb{Q} \subseteq F$ with $b > 1$.

(b) Since u is injective, the computation in the proof of part (a) shows that

$$\operatorname{supp}_{\widetilde{u}}(f_p) = \{u(\mathrm{T}_j) \mid j \in \operatorname{supp}(p)\} = \{u(T) \mid T \in \operatorname{supp}_B(p)\} = \operatorname{supp}_u(p).$$

This concludes the proof. □

Remark 5.14. While versions of Theorem 5.12 hold for any basis of polynomials satisfying a linearization relation with fixed coefficients for products (see Corollary 6.12 for a variant in the relative setting of Section 6), it is in general not easily possible to obtain corresponding versions of Corollary 5.13, i.e. sparse interpolation techniques, since bases satisfying commutativity relations are rather elusive and these conditions are not straightforward to replace. However, there are variants for other kinds of Chebyshev bases, see, e.g. Potts-Tasche [44] and Imamoglu-Kaltofen-Yang [24].

Peter and Plonka show how to view Chebyshev polynomials of the first kind as eigenfunctions of a suitable endomorphism of the space W of continuous real-valued functions on the interval $[-1, 1]$, see [37, Remark 4.6]. Thus, also the "analytic" reconstruction technique for these functions given in [44] is recast in the framework for eigenfunction sums. It is however not clear how this might be translated into a purely algebraic version.

Multivariate variants for Chebyshev polynomials of first and second kind can be found in a very recent preprint of Hubert and Singer [23].

Example 5.15. We give a toy example computation to illustrate Corollary 5.13. Let

$$f = \mathrm{y}^3 \in \mathbb{Q}[\mathrm{y}].$$

(The polynomial $f = 1/8 \cdot \mathrm{T}_3 + 1/4 \cdot \mathrm{T}_1$ has Chebyshev rank 2.) We choose $b := 2$. Then we have

$$P_2'(f) = \begin{pmatrix} 1 & 8 & 343 \\ 8 & 343 & 17576 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 343 \\ 8 & 1 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 16 & 686 \\ 16 & 344 & 17584 \end{pmatrix}$$

and

$$P_2(f) = P_2'(f) \cdot \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 16 & 344 \\ 16 & 344 & 8800 \end{pmatrix} \sim \begin{pmatrix} 1 & 8 & 172 \\ 0 & 1 & 28 \end{pmatrix}.$$

Thus,

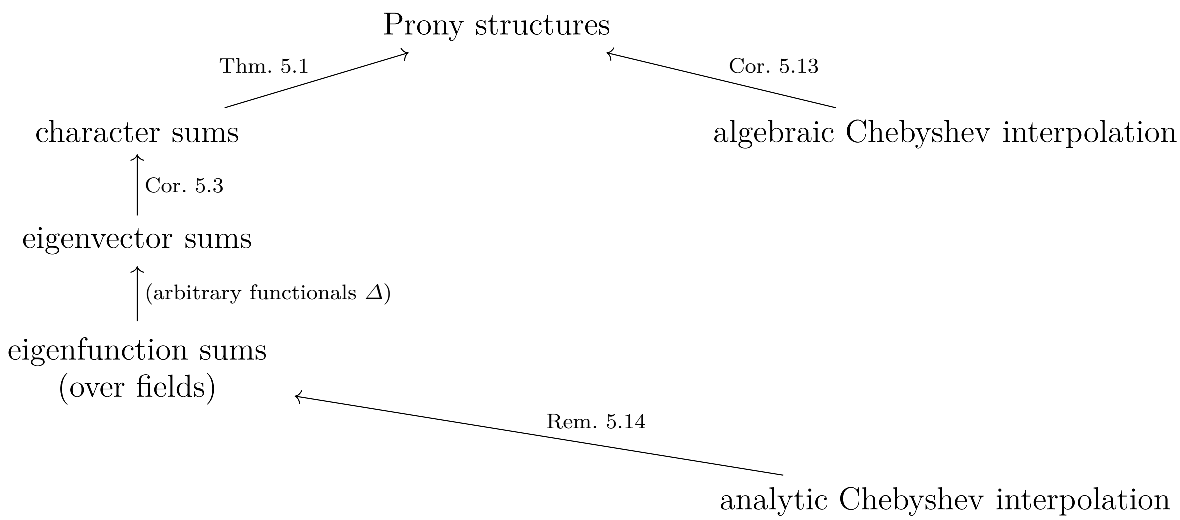
$$\ker(P_2(f)) = \langle (52, -28, 1)^\top \rangle = \langle \mathrm{x}^2 - 28\mathrm{x} + 52 \rangle = \langle (\mathrm{x} - 2)(\mathrm{x} - 26) \rangle,$$

and we recover the support of f as

$$\begin{aligned} \operatorname{supp}_B(f) &= u^{-1}(\operatorname{supp}_u(f)) = u^{-1}(\mathrm{Z}(\ker P_2(f))) = u^{-1}(\{2, 26\}) = u^{-1}(\{\mathrm{T}_1(b), \mathrm{T}_3(b)\}) \\ &= \{\mathrm{T}_1, \mathrm{T}_3\}. \end{aligned}$$

If desired, the coefficients 1/4 and 1/8 can now be easily computed by solving a 2×2-system of linear equations.

Remark 5.16. Summarizing the preceding discussion on frameworks for character [16] and eigenfunction/eigenvector sums [20, 37] and the algebraic and analytic sparse polynomial interpolation techniques w.r.t. the Chebyshev basis [32, 27] and [44], we obtain the following diagram of "inclusions".



Lakshman and Saunders remark on the possibility to "reconcile" the frameworks for character or eigenfunction sums with their algorithm for sparse polynomial interpolation w.r.t. the Chebyshev basis [32, p. 388]. As the framework of Prony structures is of a very general nature, we would not propose it as a final answer to this question. However, it can be hoped that it will be helpful in finding more particular reconciliations. See also Remark 5.14.

Remark 5.17. For sparse interpolation in various bases probabilistic results are known in the literature under the name "early termination", see for example Kaltofen-Lee [26]. In the language of the present note, there the quest is to find probabilistic estimates of the Prony index $\operatorname{ind}_P(f)$ of a polynomial f where the Prony structure P is given in similar ways as in Corollary 5.8 or Corollary 5.13. The general idea is to perform the interpolation method repeatedly on increasingly large intervals and estimate the probability of having computed the "true" interpolating polynomial in terms of the number of successive intervals with the same result and a bound for the degree of f. For more details and further refinements we refer to [26].

Early termination strategies can also be combined with sparse interpolation methods for rational functions. For details we refer to, e.g., Kaltofen-Yang [27] and Cuyt-Lee [11]. In a related direction, probabilistic methods tailored to sparse polynomial interpolation over finite fields can be found, e.g., in Arnold-Giesbrecht-Roche [2].

It would be interesting to look for generalizations of these results in the framework of Prony structures. However, in full generality this is unlikely to be fruitful, since one has to be able to make additional assumptions like degree bounds for which the Prony structures are not well-adapted.

Another potential avenue for further research could be the investigation of the computational complexity of Prony structures w.r.t. an underlying computational model, such as arithmetic circuits in polynomial identity testing. See Shpilka-Yehudayoff [50] and Saxena [48, 49] for recent surveys of this field.

We leave the search for suitable settings for the future.

Now let $A \in \mathbb{R}^{n\times n}$ be a fixed symmetric positive definite matrix. A variant of Prony's method for $\mathbb{C}$-linear combinations of the *Gaußians*

$$\mathrm{g}_{A,t} \colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \mathrm{e}^{-(x-t)^\top A(x-t)}, \quad t \in \mathbb{R}^n,$$

is proposed in Peter-Plonka-Schaback [38]. In the following we identify the underlying Prony structure. To this end, let

$$B := \{\mathrm{g}_{A,t} \mid t \in \mathbb{R}^n\} \quad \text{and} \quad V := \langle B \rangle_{\mathbb{C}}.$$

For $t \in \mathbb{R}^n$ set

$$b_{A,t} := \mathrm{e}^z = (\mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_n}) \in (\mathbb{R} \setminus \{0\})^n \text{ with } z = 2t^\top A \in \mathbb{R}^{1\times n}$$

and let

$$u\colon B \to \mathbb{R}^n, \quad \mathrm{g}_{A,t} \mapsto b_{A,t}.$$

Since A is positive definite, $\mathrm{g}_{A,t}$ obtains its unique maximum in t. This implies that u is well-defined. Also since A is positive definite, $b_{A,t} = b_{A,s}$ for $t, s \in \mathbb{R}^n$ implies that $t = s$, and thus u is injective. For the following theorem we set $\widetilde{V} := \mathrm{Exp}^n_{\mathbb{Z},u(B)}(\mathbb{C})$, $\widetilde{B} := \{\exp_b \mid b \in u(B)\}$, and $\widetilde{u}\colon \widetilde{B} \to K^n$, $\exp_b \mapsto b$. Recall that $\widetilde{B}$ is a basis of $\widetilde{V}$.

Theorem 5.18 (Prony structure for Gaußian sums)**.** *For $g \in V$ let*

$$f_g\colon \mathbb{Z}^n \to \mathbb{C}, \quad \alpha \mapsto g(\alpha) \cdot \mathrm{e}^{\alpha^\top A \alpha}.$$

Then the following holds:

(a) *For all $g \in V$ we have $f_g \in \widetilde{V}$ and $\varphi\colon V \to \widetilde{V}$, $g \mapsto f_g$, is a $\mathbb{C}$-vector space isomorphism with $\varphi(\mathrm{g}_{A,t}) = \lambda_{A,t} \cdot \exp_{b_{A,t}}$ for some $\lambda_{A,t} \in \mathbb{R} \setminus \{0\}$. In particular, B is a basis of V.*

(b) *For all $g \in V$ we have $\mathrm{supp}_{\widetilde{u}}(f_g) = \mathrm{supp}_u(g)$.*

Hence, any Prony structure on $\widetilde{V}$ (in particular the Prony structures from Theorem 4.4*), induces a Prony structure on V by the transfer principle (Lemma* 5.7*).*

Proof. (a) Note that for all $t \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}^n$ and with $\lambda_{A,t} := \mathrm{e}^{-t^\top A t} \in \mathbb{R} \setminus \{0\}$ we have

$$f_{\mathrm{g}_{A,t}}(\alpha) = \mathrm{g}_{A,t}(\alpha) \cdot \mathrm{e}^{\alpha^\top A\alpha} = \mathrm{e}^{-(\alpha-t)^\top A(\alpha-t)} \cdot \mathrm{e}^{\alpha^\top A\alpha} = \mathrm{e}^{-t^\top At} \cdot \mathrm{e}^{2t^\top A\alpha} = \lambda_{A,t} \cdot \exp_{b_{A,t}}(\alpha).$$

By definition we have $b_{A,t} \in u(B)$, and hence $\varphi(\mathrm{g}_{A,t}) = f_{\mathrm{g}_{A,t}} \in \widetilde{V}$. Since clearly $f_{\lambda g+\mu h} = \lambda f_g + \mu f_h$ for all $\lambda, \mu \in \mathbb{C}$ and $g, h \in V$, we have that $\varphi(V) \subseteq \widetilde{V}$ and φ is $\mathbb{C}$-linear. Since $\widetilde{B}$ is a $\mathbb{C}$-basis of $\widetilde{V}$, there is a unique $\mathbb{C}$-linear map $\psi\colon \widetilde{V} \to V$ with $\psi(\exp_{b_{A,t}}) = 1/\lambda_{A,t} \cdot \mathrm{g}_{A,t}$ for all $t \in \mathbb{R}^n$. Then ψ is the inverse of φ and this concludes the proof of (a).

(b) Let $g = \sum_{t\in F} \mu_t\, \mathrm{g}_{A,t}$ with finite $F \subseteq \mathbb{R}^n$ and $\mu_t \in \mathbb{C} \setminus \{0\}$. Using part (a) we obtain

$$\mathrm{supp}_{\widetilde{u}}(f_g) = \mathrm{supp}_{\widetilde{u}}\Big(\sum_{t\in F} \mu_t \lambda_{A,t} \exp_{b_{A,t}}\Big) = \{b_{A,t} \mid t \in F\} = \mathrm{supp}_u(g),$$

i.e., the assertion. □

Note that an alternative approach to the reconstruction problem in Theorem 5.18 which is based on Fourier transforms is proposed in Peter-Potts-Tasche [39].

Remark 5.19. There is a close relationship between Prony's method and Sylvester's method for computing Waring decompositions of homogeneous polynomials. Although Sylvester's method does not fit directly into our framework of Prony structures (since it is not a method to reconstruct the support of a function), one may still view it as an application of the Prony structure from Example 1.7: Given a homogeneous polynomial

$$p = \sum_{i=0}^{d} p_i \mathrm{x}^i \mathrm{y}^{d-i} \in \mathbb{C}[\mathrm{x}, \mathrm{y}],$$

of Waring rank at most r, then the matrix

$$\mathrm{C}(p) := (c_{i+j})_{\substack{i=0,\ldots,r\\ j=0,\ldots,d-r}} \in \mathbb{C}^{(r+1)\times(d-r+1)}$$

with $c_i := p_i/\binom{d}{i}$ induces a Prony structure for an exponential sum (in the sense that $\ker \mathrm{C}(p)$ identifies the support). Then this exponential sum $f_p \in \mathrm{Exp}^1(\mathbb{C})$ and its reconstruction as $f_p = \sum_{k=1}^{r} \mu_k \exp_{b_k}$ can be used to compute a Waring decomposition of p. Sylvester's method has recently been generalized to the multivariate case, cf. [5].

6. Relative Prony structures

A Prony structure on a vector space V can be seen as a tool to obtain polynomials that identify the u-support $\mathrm{supp}_u(f) \subseteq K^n$ of a given $f \in V$. Suppose that we are given a priori a set of polynomials $I \subseteq S = K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ with $\mathrm{supp}_u(f) \subseteq \mathrm{Z}(I)$. For example, one could have $K = \mathbb{R}$ and $\mathrm{supp}_u(f) \subseteq \mathbb{S}^{n-1} = \mathrm{Z}(1 - \sum_{j=1}^{n} \mathrm{x}_j^2)$. Prony structures as previously discussed do not take this additional information into account. In this section we extend Prony structures in order to take advantage of this situation.

We begin by giving appropriate variants of earlier definitions for this context.

Definition 6.1. For $Y \subseteq K^n$ let

$$K[Y] := K[\mathrm{x}]/\mathrm{I}(Y)$$

be the usual *coordinate algebra of* Y. For $D \subseteq \mathbb{N}^n$ let, as before, $\mathrm{x}^D = \{\mathrm{x}^\alpha \mid \alpha \in D\}$ and

$$\overline{\mathrm{x}^D} := \{m + \mathrm{I}(Y) \mid m \in \mathrm{x}^D\} \subseteq K[Y].$$

We denote by

$$K[Y]_D := \langle \overline{\mathrm{x}^D} \rangle_K$$

the K-subvector space of $K[Y]$ generated by $\overline{\mathrm{x}^D}$. We call $K[Y]_D$ the *coordinate space of* Y *w.r.t.* S_D.

Remark 6.2. Let $Y \subseteq K^n$ and $D \subseteq \mathbb{N}^n$. Then we have

$$K[\mathrm{x}]_D/\mathrm{I}_D(Y) \cong K[Y]_D.$$

Indeed, the K-linear map $K[\mathrm{x}]_D \to K[Y]_D$ with $\mathrm{x}^\alpha \mapsto \overline{\mathrm{x}^\alpha} = \mathrm{x}^\alpha + \mathrm{I}(Y)$ for $\alpha \in D$ is an epimorphism with kernel $\mathrm{I}_D(Y)$. In the following we identify these two K-vector spaces.

Definition 6.3. Let $D \subseteq \mathbb{N}^n$. For $X \subseteq Y \subseteq K^n$ we call

$$\mathrm{ev}_{D/Y}^X \colon K[Y]_D \to K^X, \quad p + \mathrm{I}_D(Y) \mapsto \mathrm{ev}_D^X(p) = (p(x))_{x\in X},$$

the *relative evaluation map at* X *w.r.t.* S_D *modulo* Y and

$$\mathrm{I}_{D/Y}(X) := \ker(\mathrm{ev}_{D/Y}^X)$$

the *relative vanishing space of* X *w.r.t.* S_D *modulo* Y.

Remark 6.4. Let $X \subseteq Y \subseteq K^n$ and $D \subseteq \mathbb{N}^n$, X and D finite. Since $\overline{\mathrm{x}^D}$ generates $K[Y]_D$ there is a $C \subseteq D$ such that $\overline{\mathrm{x}^C}$ is a K-basis of $K[Y]_D$. Without loss of generality, choose C such that $|\overline{\mathrm{x}^C}| = |C|$.

Observe that then the transformation matrix of $\mathrm{ev}_{D/Y}^X$ w.r.t. $\overline{\mathrm{x}^C}$ and the canonical basis of K^X is the Vandermonde matrix $\mathrm{V}_C^X = (x^\alpha)_{x\in X, \alpha\in C}$. Hence the transformation matrices of the relative evaluation map $\mathrm{ev}_{D/Y}^X$ and the "ordinary" evaluation map ev_C^X are identical.

Definition 6.5. For $J \subseteq K[Y]$ we call

$$\mathrm{Z}_Y(J) := \{y \in Y \mid \text{for all } q \in S \text{ with } q + \mathrm{I}(Y) \in J,\ q(y) = 0\}$$

the *relative zero locus of* J *w.r.t.* Y.

After these general preparations, we define relative Prony structures, which are the topic of this section. Recall that an *algebraic set* $Y \subseteq K^n$ is the zero locus of a set of polynomials, i.e., $Y = \mathrm{Z}(I)$ for some set of polynomials $I \subseteq S$. By Hilbert's basis theorem, I can always be chosen to be finite.

Definition 6.6. Given the setup of Definition 1.2, let $Y \subseteq K^n$ be an algebraic set, and suppose that

$$u(B) \subseteq Y.$$

Let $\mathcal{I} = (\mathcal{I}_d)_{d\in\mathbb{N}}$ be a sequence of finite sets and $\mathcal{H} = (\mathcal{H}_d)_{d\in\mathbb{N}}$ be a sequence of finite subsets of $\mathbb{N}^n$ such that $|\overline{\mathrm{x}^{\mathcal{H}_d}}| = |\mathcal{H}_d|$ and the vectors in the set $\overline{\mathrm{x}^{\mathcal{H}_d}}$ are linearly independent in $K[Y]$.

Let $f \in V$ and

$$P(f) = (P_d(f))_{d\in\mathbb{N}} \in \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d \times \mathcal{H}_d}.$$

We call $P(f)$ a (*relative*) *Prony structure w.r.t.* Y *for* f if for all large d one has

$$\mathrm{Z}_Y(\ker P_d(f)) = \mathrm{supp}_u(f) \quad \text{and} \quad \mathrm{I}_{\mathcal{H}_d/Y}(\mathrm{supp}_u(f)) \subseteq \ker(P_d(f)). \tag{4}$$

Here we identify $p \in \ker P_d(f) \subseteq K^{\mathcal{H}_d}$ with $\sum_{\alpha\in\mathcal{H}_d} p_\alpha \overline{\mathrm{x}^\alpha} \in K[Y]_{\mathcal{H}_d} \leq K[Y]$.

The least $c \in \mathbb{N}$ such that the conditions in (4) hold for all $d \geq c$ is called (*relative*) *Prony index w.r.t.* Y *of* f or simply P*-index w.r.t.* Y *of* f, denoted by $\mathrm{ind}_{P,Y}(f)$.

If for every $f \in V$ a relative Prony structure $P(f)$ w.r.t. Y for f is given, then we call P a (*relative*) *Prony structure w.r.t.* Y *on* V.

Remark 6.7. Over an infinite field K, Prony structures as considered before are precisely the relative Prony structures w.r.t. $Y = K^n$. This follows immediately from $K[Y] = K[\mathrm{x}]$.

We obtain a characterization of relative Prony structures analogous to one for ordinary Prony structures in Theorem 2.4.

Theorem 6.8 (Relative version of Theorem 2.4)**.** *Given the setup of Definition* 1.2*, let* $f \in V$*,* B *an* F*-basis of* V*,* $u\colon B \to K^n$ *injective,* $\mathcal{I}$ *a sequence of finite sets, and* $\mathcal{J}$ *a sequence of finite subsets of* $\mathbb{N}^n$ *with* $\mathcal{J}_d \subseteq \mathcal{J}_{d+1}$ *for all large* d *and* $\bigcup_{d\in\mathbb{N}} \mathcal{J}_d = \mathbb{N}^n$*. Let* $Y \subseteq K^n$ *be an algebraic set with*

$$\mathrm{supp}_u(f) \subseteq Y$$

and $\mathcal{H}_d \subseteq \mathcal{J}_d$ *such that* $\overline{\mathrm{x}^{\mathcal{H}_d}}$ *is a* K*-basis of* $K[Y]_{\mathcal{J}_d}$ *with* $|\overline{\mathrm{x}^{\mathcal{H}_d}}| = |\mathcal{H}_d|$*. Let*

$$Q \in \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d \times \mathcal{H}_d}.$$

Then the following are equivalent:

(i) Q *is a Prony structure w.r.t.* Y *for* f*;*
(ii) *For all large* d *there is an injective* K*-linear map* $\eta_d\colon K^{\mathrm{supp}_u(f)} \hookrightarrow K^{\mathcal{I}_d}$ *such that the diagram*

$$\begin{array}{ccc} K^{\mathcal{H}_d} & \xrightarrow{\quad Q_d \quad} & K^{\mathcal{I}_d} \\ \downarrow{\cong} & & \uparrow{\eta_d} \\ K[Y]_{\mathcal{H}_d} & \xrightarrow{\ \mathrm{ev}^{\mathrm{supp}_u(f)}_{\mathcal{H}_d/Y}\ } & K^{\mathrm{supp}_u(f)} \end{array}$$

is commutative;

(iii) *For all large d we have* $\ker(Q_d) = \mathrm{I}_{\mathcal{H}_d/Y}(\mathrm{supp}_u(f))$.

Proof. Using Remark 6.4 for the surjectivity $\mathrm{ev}_{\mathcal{H}_d/Y}^{\mathrm{supp}_u(f)}$ for all large d, the proof is analogous to the one of Theorem 2.4. □

The following theorem gives a method to obtain a relative Prony structure from an "ordinary" one. The relative Prony structure then uses smaller matrices.

Theorem 6.9. *Let P be a Prony structure on V as defined in Definition* 1.3 *and let $Y \subseteq K^n$ be an algebraic set with $u(B) \subseteq Y$. Let $\mathcal{H}_d \subseteq \mathbb{N}^n$ be such that $\overline{\mathrm{x}^{\mathcal{H}_d}}$ is a K-basis of $K[Y]_{\mathcal{J}_d} \leq K[Y]$ and $|\mathcal{H}_d| = |\overline{\mathrm{x}^{\mathcal{H}_d}}|$. Let*

$$P_{\mathcal{H}}\colon V \to \prod_{d\in\mathbb{N}} K^{\mathcal{I}_d\times\mathcal{H}_d}, \qquad f \mapsto ((P_{\mathcal{H}})_d(f))_{d\in\mathbb{N}} := (P_d(f)|_{(\mathcal{I}_d\times\mathcal{H}_d)})_{d\in\mathbb{N}}.$$

Here $P_d(f)|_{(\mathcal{I}_d\times\mathcal{H}_d)}$ is obtained from $P_d(f)$ by deleting the columns that are not in $\mathcal{H}_d$. Then $P_{\mathcal{H}}$ induces a Prony structure w.r.t. Y on V.

Proof. Let $f \in V$. By Theorem 2.4, for all large d there are injective K-linear maps $\eta_d\colon K^{\mathrm{supp}_u(f)} \hookrightarrow K^{\mathcal{I}_d}$ such that the linear map $S_{\mathcal{J}_d} \to K^{\mathcal{I}_d}$ induced by $P_d(f)$ equals $\eta_d \circ \mathrm{ev}_{\mathcal{J}_d}^{\mathrm{supp}_u(f)}$.

It is easy to see that then the linear map $S_{\mathcal{H}_d} \to K^{\mathcal{I}_d}$ induced by $(P_{\mathcal{H}})_d(f)$ equals $\eta_d \circ \mathrm{ev}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)}$ (for all large d). Recall that the matrix of $\mathrm{ev}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)}$ equals $\mathrm{V}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)}$. Thus, we have $(P_{\mathcal{H}})_d(f) = E_d\cdot\mathrm{V}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)}$ where E_d denotes the matrix of η_d. Hence, by Remark 6.4 we have that the linear map $K[Y]_{\mathcal{H}_d} \to K^{\mathcal{I}_d}$ induced by $(P_{\mathcal{H}})_d(f)$ equals $\eta_d \circ \mathrm{ev}_{\mathcal{H}_d/Y}^{\mathrm{supp}_u(f)}$. By the direction "(ii) ⇒ (i)" of Theorem 6.8 we are done. □

Corollary 6.10 (Relative version of Theorem 4.4 (a))**.** *Let $V := \mathrm{Exp}_Y^n(F)$ with an algebraic set $Y \subseteq K^n$. For appropriately chosen sequences $\mathcal{I}$ and $\mathcal{J}$,*

$$\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f) = (f(\alpha+\beta))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{J}_d}} \in K^{\mathcal{I}_d\times\mathcal{J}_d}$$

induces a Prony structure on V according to Theorem 4.4 (a). *Let $\mathcal{H}_d \subseteq \mathcal{J}_d$ be such that $\overline{\mathrm{x}^{\mathcal{H}_d}}$ is a K-basis of $K[Y]_{\mathcal{J}_d}$. Then*

$$\mathrm{H}_{\mathcal{I},\mathcal{H},d}(f) = (f(\alpha+\beta))_{\substack{\alpha\in\mathcal{I}_d\\ \beta\in\mathcal{H}_d}} \in K^{\mathcal{I}_d\times\mathcal{H}_d}$$

induces a Prony structure w.r.t. Y on V.

Proof. This is immediate by applying Theorem 6.9 to Theorem 4.4 (a). □

Remark 6.11.

(a) An analogous result to Corollary 6.10 holds for the Toeplitz Prony structure on $V = \mathrm{Exp}_Y^n(F)$ for an (algebraic) set $Y \subseteq (K \setminus \{0\})^n$.

(b) For $V = \mathrm{Exp}_Y^n(F)$ as in Corollary 6.10 a more efficient result is possible as follows.
As a matrix, $\mathrm{H}_{\mathcal{I},\mathcal{H},d}(f)$ is obtained by "deleting columns" from $\mathrm{H}_{\mathcal{I},\mathcal{J},d}(f)$. By the proof of Theorem 4.4 (a), the linear map $S_{\mathcal{H}_d} \to K^{\mathcal{I}_d}$ induced by $\mathrm{H}_{\mathcal{I},\mathcal{H},d}(f)$ equals $\eta_{\mathcal{I},d} \circ \mathrm{ev}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)}$ with $\eta_{\mathcal{I},d} := (\mathrm{ev}_{\mathcal{I}_d}^{\mathrm{supp}_u(f)})^\top \circ C_f$. Thus, we may also pass to $\eta_{\mathcal{H},d} := (\mathrm{ev}_{\mathcal{H}_d}^{\mathrm{supp}_u(f)})^\top \circ C_f$, since also $\eta_{\mathcal{H},d}$ is injective for all large d. Hence also $\mathrm{H}_{\mathcal{H},\mathcal{H},d}(f) = (f(\alpha+\beta))_{\alpha,\beta\in\mathcal{H}_d}$ induces a Prony structure w.r.t. Y on $V = \mathrm{Exp}_Y^n(F)$.

While Theorem 6.9 yields a general recipe to construct relative Prony structures from "ordinary" ones, in concrete situations it can be possible to achieve better results. We end the section with one such example, recasting the main result of [30] in the context of relative Prony structures. Let $K = \mathbb{R}$, $S = \mathbb{R}[\mathrm{x}_1, \dots, \mathrm{x}_n]$, and

$$Y := \mathbb{S}^{n-1} = \mathrm{Z}\Big(1 - \sum_{j=1}^{n} \mathrm{x}_j^2\Big) = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\} \subseteq \mathbb{R}^n.$$

Consider the $\mathbb{R}$-vector space

$$\mathrm{SH}_{\le d} := \mathbb{R}[\mathbb{S}^{n-1}]_{\le d} = S_{\le d}/\mathrm{I}_{\le d}(\mathbb{S}^{n-1}) \cong \{p|_{\mathbb{S}^{n-1}} \mid p \in S_{\le d}\}.$$

Let $\Delta\colon S \to S$, $p \mapsto \sum_{j=1}^n \partial_j^2(p)$, denote the *Laplace operator*. The elements of $\ker(\Delta)$ are called *harmonic*.

Let harmH_k be the $\mathbb{R}$-vector space generated by the restrictions $p|_{\mathbb{S}^{n-1}}$ of harmonic homogeneous polynomials $p \in S_k$ of degree k to the sphere, usually called the space of *spherical harmonics*. Using Gallier-Quaintance [18, Theorem 6.13, discussion after Definition 6.15] it is easy to see that one has the decomposition (as vector spaces)

$$\mathrm{SH}_{\le d} \cong \bigoplus_{k=0}^{d} \mathrm{harmH}_k.$$

For $k = 0, \dots, d$, let $H_k = (y_k^1, \dots, y_k^{d_k})$ be an $\mathbb{R}$-basis of harmH_k. Hence $H_{\le d} := \bigcup_{k=0}^d H_k$ is a basis of $\mathrm{SH}_{\le d}$. For $x \in \mathbb{S}^{n-1}$ let

$$h_x\colon \{(k, \ell) \mid k \in \mathbb{N},\, \ell = 1, \dots, d_k\} \to \mathbb{R}, \quad (k, \ell) \mapsto y_k^\ell(x).$$

For finite $X \subseteq \mathbb{S}^{n-1}$ let $W_{\le d}^X$ be the matrix of $\mathrm{ev}^X_{\mathcal{T}_d/\mathbb{S}^{n-1}}$ w.r.t. $H_{\le d}$ and the basis of $\mathbb{R}^X$.

Corollary 6.12 (Relative Prony structure for spherical harmonic sums)**.** *Let* $B := \{h_x \mid x \in \mathbb{S}^{n-1}\}$, $V := \langle B\rangle_{\mathbb{R}}$, *and* $u\colon B \to \mathbb{R}^n$, $h_x \mapsto x$. *For* $f \in V$, $f = \sum_{x \in \mathrm{supp}_u(f)} f_x h_x$, $f_x \in \mathbb{R} \setminus \{0\}$, *let* $C_f = (f_x \mathrm{e}_x)_{x \in X}$ *and*

$$\widetilde{\mathrm{H}}_d(f) = (W_{\le d}^{\mathrm{supp}_u(f)})^\top \cdot C_f \cdot W_{\le d}^{\mathrm{supp}_u(f)}.$$

Then the function

$$\widetilde{\mathrm{H}}\colon V \to \prod_{d \in \mathbb{N}} \mathbb{R}^{H_{\le d} \times H_{\le d}}, \quad f \mapsto (\widetilde{\mathrm{H}}_d(f))_{d \in \mathbb{N}},$$

induces a relative Prony structure w.r.t. $\mathbb{S}^{n-1}$ *on* V.

Proof. This follows from Kunis-Möller-von der Ohe [30, Section 3.3, Theorem 3.14]. □

Remark 6.13. Observe that by [30, Theorem 3.14], the matrix $\widetilde{\mathrm{H}}_d(f)$ can be computed solely from $\Theta(d^{n-1})$ evaluations of f. One may also use Corollary 6.10 or even Remark 6.11 (b) to get a Prony structure w.r.t. $\mathbb{S}^{n-1}$ on $\mathrm{SH}_{\le d}$. The matrices so obtained have the same number of columns or the same size as the ones in Corollary 6.12, respectively. But then the number $|\mathcal{H}_d + \mathcal{H}_d|$ of used evaluations is not in general in $\Theta(d^{n-1})$.

7. Maps between Prony structures

In Section 5 we witnessed instances of Prony structures transferring from one vector space to another, such as from spaces of exponential sums to spaces of polynomials or Gaußian sums with their respective bases. We take these observations as motivation to consider structure preserving maps between Prony structures. For notational simplicity, whenever we say that P is a Prony structure, we mean that P is a Prony structure on an F-vector space V with basis B w.r.t. an injection $u\colon B \to K^n$. Similarly, when P' is a Prony structure, then this means that P' is a Prony structure on an F'-vector space V' with basis B' w.r.t. an injection $u'\colon B' \to (K')^{n'}$.

The following is natural definition of structures preserving maps between Prony structures.

Definition 7.1. Let P and P' be Prony structures on V and V', respectively. Let

- $\iota\colon F \to F'$ be a field homomorphism (turning V' into an F-vector space),
- $\varphi\colon V \to V'$ be an F-vector space homomorphism, and
- $\mu\colon P(V) \to P'(V')$ be a function, where $P(V) = \{(P_d(f))_{d\in\mathbb{N}} \mid f \in V\}$.

Then $\psi := (\iota, \varphi, \mu)$ is called *map of Prony structures from P to P'*, abbreviated as *Prony map* in the following, written $\psi\colon P \to P'$, if the inclusion

$$\varphi(B) \subseteq B'$$

holds and the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{P} & P(V) \\ \big\downarrow{\scriptstyle\varphi} & & \big\downarrow{\scriptstyle\mu} \\ V' & \xrightarrow{P'} & P'(V') \end{array}$$

Remark 7.2. Our notation should not be confused with a similar definition in Batenkov-Yomdin [3] where certain moment maps are considered.

One might expect a map between K^n and $(K')^{n'}$ in the definition of Prony map (that is compatible with the other data). However, if P and P' are Prony structures and $\psi = (\iota, \varphi, \mu)\colon P \to P'$ is a Prony map then, since u is injective, there is always a function

$$\varrho_\psi\colon u(B) \to u'(B'), \quad \ell \mapsto (u' \circ \varphi)(u^{-1}(\ell)),$$

that maps elements of $u(B) \subseteq K^n$ to elements of $u'(B') \subseteq (K')^{n'}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & B' \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle u'} \\ u(B) & \xrightarrow{\varrho_\psi} & u'(B') \end{array}$$

Clearly, ϱ_ψ is injective if and only if φ is injective.

Remark 7.3. Let $\mathcal{P} = (\mathcal{O}, \mathrm{Hom}, \mathrm{id}, \circ)$ be defined as follows.

- $\mathcal{O} := \{P \mid P \text{ Prony structure}\}$ is the class of all Prony structures.
- For $P, P' \in \mathcal{O}$, $\mathrm{Hom}(P, P') := \{\psi \mid \psi\colon P \to P' \text{ Prony map}\}$ is the set of all Prony maps from P to P'.
- For $P \in \mathcal{O}$, let
$$\mathrm{id}_P := (\mathrm{id}_F, \mathrm{id}_V, \mathrm{id}_{P(V)}).$$
- For $P, P', P'' \in \mathcal{O}$, $\psi = (\iota, \varphi, \mu) \in \mathrm{Hom}(P, P')$, and $\psi' = (\iota', \varphi', \mu') \in \mathrm{Hom}(P', P'')$, let
$$\psi' \circ \psi := ((\iota' \circ \iota), (\varphi' \circ \varphi), (\mu' \circ \mu)).$$

It is straightforward to show that $\mathcal{P}$ is a category (cf., e.g., [34, 1]). We call $\mathcal{P}$ the *category of Prony structures*. It would be interesting to get insights from this point of view.

Example 7.4 (Sparse polynomial interpolation)**.** Let the notation and assumptions be as in Corollary 5.8, and moreover let $\iota := \mathrm{id}_F$ be the identity map on F. Note that

$$Q_P(V) = \{Q_P(p) \mid p \in V\} = \{P(f_p) \mid p \in V\} \subseteq P(\widetilde{V}).$$

So we choose $\mu\colon Q_P(V) \hookrightarrow P(\widetilde{V})$ to be the inclusion map. Then $\psi := (\iota, \varphi, \mu)$ is a Prony map from Q_P to P. Indeed, easy computations show that $\varphi\colon V \to \widetilde{V}$ is a vector space homomorphism and that $\mu \circ Q_P = P \circ \varphi$.

Example 7.5 (Projection methods)**.** For $n \in \mathbb{N}$ let $V_n := \mathrm{Exp}^n_{K^n}(F)$. Let H_n be the Prony structure derived from Theorem 4.4 (a).

For a fixed $\alpha \in \mathbb{N}^n$ let

$$\varphi_\alpha\colon V_n \to V_1, \quad f \mapsto f_\alpha,$$

where

$$f_\alpha\colon \mathbb{N} \to K, \quad k \mapsto f(k \cdot \alpha).$$

It is easy to see that $f_\alpha \in V_1$ and hence φ is well-defined. Furthermore, let

$$\mu_\alpha\colon \mathrm{H}_n(V_n) \to \mathrm{H}_1(V_1), \quad \mathrm{H}_n(f) \mapsto \mathrm{H}_1(f_\alpha).$$

Then $\psi_\alpha := (\mathrm{id}_F, \varphi_\alpha, \mu_\alpha)$ is a Prony map from H_n to H_1.

Also note that $\mathrm{H}_{1,d}(f_\alpha) = (\mathrm{H}_{n,d}(f)_{\beta,\gamma})_{\beta,\gamma \in \mathcal{J}_{1,d}\cdot\alpha}$.

Proof. It is easy to verify that φ_α is F-linear. Furthermore, for every $b \in K^n$ we have

$$\varphi_\alpha(\exp_b) = \exp_{b^\alpha},$$

hence $\varphi_\alpha(B_n) \subseteq B_1$. The identity $\mathrm{H}_1 \circ \varphi_\alpha = \mu_\alpha \circ \mathrm{H}_n$ holds by the definitions.

Finally, let $f \in V_n$ and $d \in \mathbb{N}$. We have

$$\begin{aligned}\mathrm{H}_{1,d}(f_\alpha) &= (f_\alpha(k+\ell))_{k,\ell \in \mathcal{J}_{1,d}} = (f((k+\ell)\alpha))_{k,\ell \in \mathcal{J}_{1,d}} = (f(k\alpha + \ell\alpha))_{k,\ell \in \mathcal{J}_{1,d}} \\ &= (f(\beta+\gamma))_{\beta,\gamma \in \mathcal{J}_{1,d}\cdot\alpha} = (\mathrm{H}_{n,d}(f)_{\beta,\gamma})_{\beta,\gamma \in \mathcal{J}_{1,d}\cdot\alpha},\end{aligned}$$

which concludes the proof. □

Remark 7.6. For the Prony structures T_n and H_n from Theorem 4.4 (b), (c) Prony maps $\mathrm{T}_n \to \mathrm{T}_1$ and $\mathrm{H}_n \to \mathrm{H}_1$ can be constructed analogously to Example 7.5.

Example 7.7. There is a Prony map $\psi = (\iota, \varphi, \mu)\colon \mathrm{T} \to \mathrm{H}$ given by $\iota = \mathrm{id}_F$, $\varphi = \mathrm{id}_{\mathrm{Exp}^n_Y(F)}$, and $\mu(\mathrm{T}(f)) = \mathrm{H}(f)$. Note that μ is well-defined since all the coefficients of $\mathrm{H}_d(f) = (f(\alpha+\beta))_{\alpha,\beta \in \mathcal{J}_d}$ appear in the matrix $\mathrm{T}_e(f) = (f(\alpha-\beta))_{\beta,\alpha \in \mathcal{J}_e}$ for some $e \in \mathbb{N}$.

Example 7.8 (Gaußian sums)**.** Let the notation and assumptions be as in Theorem 5.18 and let $\widetilde{C} := \varphi(B) = \{\lambda_{A,t} \cdot \exp_{b_{A,t}} \mid t \in \mathbb{R}^n\}$. Clearly, $\widetilde{C}$ is a basis of $\widetilde{V}$. Let $\widetilde{v}\colon \widetilde{C} \to \mathbb{R}^n$, $\varphi(b) \mapsto \widetilde{u}(b)$ and let P be any Prony structure on $\widetilde{V}$ w.r.t. $\widetilde{v}$. Let $\iota\colon \mathbb{C} \to \mathbb{C}$ be the identity map. It is again easy to see that $Q_P(V) \subseteq P(\widetilde{V})$. Thus, let $\mu\colon Q_P(V) \hookrightarrow P(\widetilde{V})$ be the inclusion map. Then $\psi := (\iota, \varphi, \mu)\colon Q_P \to P$ is a Prony map. Indeed, we have already seen in Theorem 5.18 that $\varphi\colon V \to \widetilde{V}$ is a $\mathbb{C}$-vector space isomorphism. By the definitions, we have $\varphi(B) \subseteq \widetilde{C}$ and the diagram

$$\begin{array}{ccc} V & \xrightarrow{Q_P} & Q_P(V) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mu} \\ \widetilde{V} & \xrightarrow{P} & P(\widetilde{V}) \end{array}$$

is commutative.

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