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Extended Abstracts

Editors

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PREFACE

In the fall of 2007 the University of Osnabrück advertized a competition for the support of graduate colleges. The algebra and topology groups of the Institute of Mathematics decided to participate and to apply for the support of a graduate college *Kombinatorische Strukturen in Algebra und Geometrie*. In April 2008 we were among the winners of the competition, and in April 2009 the five PhD students of the college started their work.

The *conference on Combinatorial Structures in Algebra and Topology* was held for the celebration of the official opening of the college. The lectures of internationally renowned speakers invited and the short communications of young mathematicians gave an excellent overview of the field and demonstrated the manifold cross-connections between algebra and topology.

Simone Böttger and Christof Söger were very helpful in the organization of the conference. In addition, Simone Böttger very patiently took care of the technical preparation of this volume.

The organizers cordially thank Marianne Gausmann and Sabine Jones for the smooth organization and for providing a comfortable atmosphere.

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Higher complements of combinatorial sphere arrangements

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We introduce higher complements of real hyperplane and sphere arrangements and show that they have the homotopy type of an explicitly given finite CW-complex. The formulas extend those given by Salvetti for complements of complexified real hyperplane arrangements. We end with a conjectural application to structured analogs of the little k-cubes operad of Boardman and Vogt.

1. HYPERPLANE ARRANGEMENTS

A (central) *hyperplane arrangement* $\mathscr{A}$ in euclidean space V is a finite family $(H_\alpha)_{\alpha\in\mathscr{A}}$ of hyperplanes of V containing the origin. The arrangement is *essential* if its center $\bigcap_{\alpha\in\mathscr{A}} H_\alpha$ is trivial.

The *complement* $\mathscr{M}(\mathscr{A}) = V\backslash(\bigcup_{\alpha\in\mathscr{A}} H_\alpha)$ decomposes into path components, called *chambers* (or *topes*): $\mathscr{C}_{\mathscr{A}} = \pi_0(\mathscr{M}(\mathscr{A}))$.

Denote by s_α the *reflection* with respect to H_α. If $(H_\alpha)_{\alpha\in\mathscr{A}}$ is stable under s_β for all $\beta\in\mathscr{A}$, the arrangement is called a *Coxeter arrangement*. We write $\mathscr{A}=\mathscr{A}_W$ where W is the subgroup of the orthogonal $O_n(\mathbb{R})$ generated by the reflections $s_\alpha, \alpha\in\mathscr{A}$. This notation is justified by

Proposition 1.1 (Coxeter, Tits)**.** *There is a one-to-one correspondence between essential Coxeter arrangements $\mathscr{A}_W$ and finite Coxeter groups W. The latter are classified by their Coxeter diagrams.*

The Coxeter group W acts simply transitively on $\mathscr{C}_{\mathscr{A}_W}$.

Definition 1.2. *The k-th* complement *of a hyperplane arrangement $\mathscr{A}$ is*

$$\mathscr{M}_k(\mathscr{A}) = V^k\backslash \textstyle\bigcup_{\alpha\in\mathscr{A}}(H_\alpha)^k.$$

Example (the braid arrangement). Let $V=\mathbb{R}^n$, $\mathscr{A}=(H_{ij})_{1\le i<j\le n}$ where $H_{ij}=\{x\in\mathbb{R}^n | x_i = x_j\}$. This is the Coxeter arrangement $\mathscr{A}_{\mathfrak{S}_n}$ for the *symmetric group* $\mathfrak{S}_n$. The center is $\mathbb{R}.(1,\dots,1)$. The higher complements are *configuration spaces*:

$$\mathscr{M}_k(\mathscr{A}_{\mathfrak{S}_n}) = F(\mathbb{R}^k,n) = \{(x_1,\dots,x_n)\in\mathbb{R}^{kn} | x_i\neq x_j\}.$$

Proposition 1.3 (Brieskorn [5])**.** *The fundamental group of the complement of a complexified arrangement $\mathscr{A}_W$ is the* pure Artin group *of W:*

$$\pi_1(\mathscr{M}_2(\mathscr{A}_W)) = Ker(A_W\to W).$$

Theorem 1.4 (Deligne [6])**.** *For any simplicial hyperplane arrangement $\mathscr{A}$, the complement of the complexified arrangement $\mathscr{M}_2(\mathscr{A})$ is aspherical.*

Observe that Coxeter arrangements are simplicial; in particular we get $\mathscr{M}_2(\mathscr{A}_W) = K(A_W, 1)$. Observe also that for $k > 2$, the k-th complement $\mathscr{M}_k(\mathscr{A})$ is always $(k-1)$-connected, but its homotopy is in general not concentrated in degree k as in the case $k = 2$.

Purpose of the talk: Describe a *finite cell complex* $\mathscr{S}^{(k)}_{\mathscr{A}}$ of the homotopy type of $\mathscr{M}_k(\mathscr{A})$ extending the following known constructions:

- Fox-Neuwirth [11] and Milgram [15] construct $\mathscr{S}^{(k)}_{\mathscr{A}_{\mathfrak{S}_n}}$ for any k;
- Salvetti [20] constructs $\mathscr{S}^{(2)}_{\mathscr{A}}$ for any arrangement $\mathscr{A}$.

Theorem 1.5 (Randell[18], Dimca-Papadima[7], Salvetti-Settepanella[21])**.** *The complement of a complex hyperplane arrangement admits a* minimal CW-structure*. The minimal CW-structure of $\mathscr{M}_2(\mathscr{A})$ derives from the Salvetti complex $\mathscr{S}^{(2)}_{\mathscr{A}}$ through combinatorial Morse theory.*

Remark 1.6 (Gel'fand-Rybnikov [12]). The Salvetti complex $\mathscr{S}^{(2)}_{\mathscr{A}}$ depends only on the *oriented matroid* $\mathscr{F}_{\mathscr{A}}$ of the hyperplane arrangement $\mathscr{A}$.

2. Oriented matroids

Orient a hyperplane arrangement $\mathscr{A}$ in V, by choosing for each H_α two half-spaces $H^{\pm}_\alpha$ such that $H^+_\alpha \cap H^-_\alpha = H_\alpha$ and $H^+_\alpha \cup H^-_\alpha = V$. Then each point $x \in V$ defines a sign vector $sgn_x \in \{0,\pm\}^{\mathscr{A}}$ by

$$sgn_x(\alpha) = \begin{cases} 0 & \text{if } x \in H_\alpha; \\ \pm & \text{if } x \in H^{\pm}_\alpha \backslash H_\alpha. \end{cases}$$

The *oriented matroid* $\mathscr{F}_{\mathscr{A}} \subset \{0,\pm\}^{\mathscr{A}}$ is the set of all such sign vectors $sgn_x, x \in V$, equipped with the *partial order* induced from the product order on $\{0,\pm\}^{\mathscr{A}}$ where $0 < +$ and $0 < -$.

Each $P \in \mathscr{F}_{\mathscr{A}}$ defines a *facet* $c_P = \{x \in V | sgn_x = P\}$. The facets are convex subsets of V, open in their closure. By definition,

$$\bar{c}_P \subseteq \bar{c}_Q \text{ in } V \text{ iff } P \leq Q \text{ in } \mathscr{F}_{\mathscr{A}}.$$

The unit-sphere S_V gets a *CW-structure* with cell poset $\mathscr{F}_{\mathscr{A}} \backslash \{0\}$. In particular, $\mathscr{F}_{\mathscr{A}}$ is a *CW-poset* in the sense of Björner [4]. More is true: $\mathscr{F}_{\mathscr{A}} \backslash \{0\}$ is a so-called *PL-sphere*, i.e. its nerve is combinatorially equivalent to the boundary of a simplex. In particular, the opposite poset $\mathscr{F}^{\mathrm{op}}_{\mathscr{A}}$ is also a CW-poset.

For $P, Q \in \mathscr{F}_{\mathscr{A}}$ we define a sign vector $PQ \in \{0,\pm\}^{\mathscr{A}}$ by

$$(PQ)(\alpha) = \begin{cases} P(\alpha) & \text{if } P(\alpha) \neq 0; \\ Q(\alpha) & \text{if } P(\alpha) = 0. \end{cases}$$

The subset $\mathscr{F}_{\mathscr{A}} \subset \{0,\pm\}^{\mathscr{A}}$ of sign vectors of the arrangement $\mathscr{A}$ fulfills the following defining properties of an *oriented matroid*:

(1) $0 \in \mathscr{F}_{\mathscr{A}}$;
(2) $P \in \mathscr{F}_{\mathscr{A}}$ implies $-P \in \mathscr{F}_{\mathscr{A}}$;

(3) $P,Q \in \mathscr{F}_{\mathscr{A}}$ implies $PQ \in \mathscr{F}_{\mathscr{A}}$;
(4) Any $\alpha \in \mathscr{A}$ which separates $P,Q \in \mathscr{F}_{\mathscr{A}}$ supports an $R \in \mathscr{F}_{\mathscr{A}}$ such that $R(\beta) = (PQ)(\beta) = (QP)(\beta)$ for non separating $\beta \in \mathscr{A}$.

Here, α is said to *separate* P,Q if $P(\alpha)Q(\alpha) = -1$, and α is said to *support* R if $R(\alpha) = 0$.

A *sphere arrangement* in V is a collection $(S_\alpha)_{\alpha\in\mathscr{A}}$ of *centrally symmetric* codimension one subspheres of the unit-sphere S_V such that

(1) The closures $S_\alpha^\pm$ of the two components of $S_V \backslash S_\alpha$ are balls;
(2) any intersection of the $S_\alpha^\pm$ is either a ball, a sphere or empty.

A sphere arrangement $(S_\alpha)_{\alpha\in\mathscr{A}}$ defines an oriented matroid $\mathscr{F}_{\mathscr{A}} \subset \{0,\pm\}^{\mathscr{A}}$ with respect to the *pseudohyperplane* arrangement $(\mathbb{R}.S_\alpha)_{\alpha\in\mathscr{A}}$.

Theorem 2.1 (Folkman-Lawrence '78, Edmonds-Mandel '78). *Any simple oriented matroid $\mathscr{F}_{\mathscr{A}} \subset \{0,\pm\}^{\mathscr{A}}$ is the oriented matroid of an essentially unique sphere arrangement in $V = \mathbb{R}^{\mathrm{rk}(\mathscr{F}_{\mathscr{A}})}$.*

Definition 2.2. *The k-th complement of a sphere arrangement $(S_\alpha)_{\alpha\in\mathscr{A}}$ in euclidean space V is given by*

$$\mathscr{M}_k(\mathscr{A}) = V^k \backslash \bigcup_{\alpha\in\mathscr{A}} (\mathbb{R}.S_\alpha)^k \simeq \underbrace{S_V * \cdots * S_V}_{k} \backslash \bigcup_{\alpha\in\mathscr{A}} \underbrace{S_\alpha * \cdots * S_\alpha}_{k}$$

where $X * Y$ denotes the topological join of X and Y.

3. Higher Salvetti complexes

Throughout, $\mathscr{A}$ denotes a hyperplane (or sphere) arrangement in euclidean space V. All our constructions depend only on the oriented matroid $\mathscr{F}_{\mathscr{A}}$ of the arrangement $\mathscr{A}$. For each poset X, we denote by $|X|$ the geometric realization of the simplicial nerve of X. The symbol $\simeq$ denotes homotopy equivalence.

The chamber system $\mathscr{C}_{\mathscr{A}}$ is the *discrete* subposet of $\mathscr{F}_A$ consisting of the *maximal* facets of $\mathscr{F}_{\mathscr{A}}$. In particular, $|\mathscr{C}_{\mathscr{A}}| \simeq \mathscr{M}(\mathscr{A})$. It is important that the oriented matroid construction is compatible with cartesian product in the following sense:

$\mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{A}} = \mathscr{F}_{\mathscr{A}\oplus\mathscr{A}}$ where $\mathscr{A} \oplus \mathscr{A} = (\mathscr{A} \times V) \cup (V \times \mathscr{A})$ in $V \times V$.

Definition 3.1 (Orlik [16]). $\mathscr{C}_{\mathscr{A}}^{(2)} := \{(P,Q) \in \mathscr{F}_{\mathscr{A}} \times \mathscr{F}_{\mathscr{A}} \,|\, PQ \in \mathscr{C}_{\mathscr{A}}\}^{\mathrm{op}}$

Observe that $(P,Q) \notin \mathscr{C}_{\mathscr{A}}^{(2)}$ if and only if $\exists \alpha \in \mathscr{A} : P(\alpha) = Q(\alpha) = 0$.

Since $\mathscr{F}_{\mathscr{A}}$ is a CW-poset [4], its realization $|\mathscr{F}_{\mathscr{A}}|$ may be identified with the simplicial complex obtained from the barycentric subdivision of a regular CW-complex. Simplicial complexes possess the following explicit form of *excision*: For subcomplexes K_1, K_2 of a simplicial complex L such that $Vert(L) = Vert(K_1) \sqcup Vert(K_2)$, one has: $|L| \backslash |K_1| \simeq |K_2|$. All this put together yields

Proposition 3.2 (Orlik [16]). $|\mathscr{C}_{\mathscr{A}}^{(2)}| \simeq \mathscr{M}_2(\mathscr{A})$.

Definition 3.3 (Salvetti [20]). *$\mathscr{S}^{(2)}_{\mathscr{A}} = \{(P,C) \in \mathscr{F}_{\mathscr{A}} \times \mathscr{C}_{\mathscr{A}} \,|\, P \leq C\}$ where $(P,C) \geq (P',C')$ iff $P \leq P'$ and $P'C = C'$.*

Theorem 3.4 (Salvetti [20], Arvola [1]). $|\mathscr{S}^{(2)}_{\mathscr{A}}| \simeq \mathscr{M}_2(\mathscr{A})$.

Proof. — The map $(P,Q) \mapsto (P,PQ)$ is a homotopy equivalence of posets $\mathscr{C}^{(2)}_{\mathscr{A}} \xrightarrow{\sim} \mathscr{S}^{(2)}_{\mathscr{A}}$. Indeed, by Quillen's [17] Theorem A, it suffices to show that the homotopy fibers $c_{(P,C)} = \{Q \in \mathscr{F}_{\mathscr{A}} \,|\, PQ \leq C\}$ are contractible. For $\mathscr{A}_{|P|} = \{\alpha \in \mathscr{A} \,|\, P(\alpha) = 0\}$ we get the identification

$$c_{(P,C)} = \{Q \in \mathscr{F}_{\mathscr{A}} \,|\, Q(\alpha) \leq C(\alpha),\, \alpha \in \mathscr{A}_{|P|}\}.$$

Thus, $c_{(P,C)}$ maps to the closure of a chamber in $\mathscr{F}_{\mathscr{A}/|P|}$ under the canonical homotopy equivalence $\mathscr{F}_{\mathscr{A}} \backslash \mathscr{F}_{|P|} \simeq \mathscr{F}_{\mathscr{A}/|P|}$. □

Savetti [20] shows more: he exhibits $|\mathscr{S}^{(2)}_{\mathscr{A}}|$ as a deformation retract of $\mathscr{M}_2(\mathscr{A})$. The proof-sketch above is inspired by Arvola's [1] proof and has the advantage of being easily extendable to higher complements.

More concrete proofs use suitable covers of $\mathscr{M}_2(\mathscr{A})$. For sake of completeness we present briefly two different (somehow dual) covers, induced by a common *stratification* of $\mathscr{M}_2(\mathscr{A})$:

Let $st_{(P,C)} = \{(x_1,x_2) \in V \times V \,|\, x_1 \in c_P; x_2 \in c_C \mod |P|\}$. These are *convex* subsets of $\mathscr{M}_2(\mathscr{A})$, open in their closure. They define a *stratification* of $\mathscr{M}_2(\mathscr{A})$ labelled by $\mathscr{S}^{(2)}_{\mathscr{A}}$ such that

$$\overline{st}_{(P,C)} \subseteq \overline{st}_{(P',C')} \text{ in } \mathscr{M}_2(\mathscr{A}) \text{ iff } (P,C) \geq (P',C') \text{ in } \mathscr{S}^{(2)}_{\mathscr{A}}.$$

The intersection of two closed strata is a union of closed strata. Any closed stratum is contractible. Moreover, inclusions of closed strata are closed cofibrations. This implies by a nowadays classical homotopy colimit argument (cf. Reedy [19]) that $\mathscr{M}_2(\mathscr{A}) \simeq |\mathscr{S}^{(2)}_{\mathscr{A}}|$.

Equivalently, let $V_{(P,C)} = \bigcup_{(P,C) \geq (P',C')} st_{(P',C')}$. This defines an *open cover* of $\mathscr{M}_2(\mathscr{A})$ used already by Deligne [6]. The open subsets $V_{(P,C)}$ are contractible and $V_{(P,C)} \subseteq V_{(P',C')}$ iff $(P,C) \leq (P',C')$. Moreover, each intersection $V_{(P,C)} \cap V_{(P',C')}$ is a union of $V_{(P'',C'')}$'s. Another homotopy colimit argument (cf. McCord [14]) yields $\mathscr{M}_2(\mathscr{A}) \simeq |\mathscr{S}^{(2)}_{\mathscr{A}}|$.

Definition 3.5. *We define the following posets:*

$$\mathscr{C}^{(k)}_{\mathscr{A}} = \{(P_1,\dots,P_k) \in (\mathscr{F}_{\mathscr{A}})^k \,|\, P_1 \cdots P_k \in \mathscr{C}_{\mathscr{A}}\}^{\mathrm{op}}$$
$$\mathscr{S}^{(k)}_{\mathscr{A}} = \{(P_1,\dots,P_{k-1},C) \in (\mathscr{F}_A)^{k-1} \times \mathscr{C}_{\mathscr{A}} \,|\, P_1 \leq \cdots \leq P_{k-1} \leq C\}$$

where $(P_1,\dots,P_{k-1},C) \geq (P'_1,\dots,P'_{k-1},C')$ iff $\forall i : P_i \leq P'_i \wedge P'_iC = C'$.

Theorem 3.6. *$|\mathscr{C}^{(k)}_{\mathscr{A}}| \simeq \mathscr{M}_k(\mathscr{A})$ and a homotopy equivalence of posets $\mathscr{C}^{(k)}_{\mathscr{A}} \xrightarrow{\sim} \mathscr{S}^{(k)}_{\mathscr{A}}$ is given by $(P_1,\dots,P_k) \mapsto (P_1, P_1P_2, \dots, P_1P_2 \cdots P_k)$.*

Proof. — The homotopy fibers $c_{(P_1,\dots,P_{k-1},C)}$ are homotopy colimits over $\{Q \,|\, P_1Q \leq P_2\}$ of homotopy fibers $c_{(P_2,\dots,P_{k-1},C)}$. □

Definition 3.7. *For $C \in \mathscr{C}_{\mathscr{A}}$, a function $\mu : \mathscr{A} \to \{0,1,\dots,k-1\}$ is* C-admissible *iff* $\exists (P_1,\dots,P_{k-1},C) \in \mathscr{S}^{(k)}_{\mathscr{A}} : \mu(\alpha) = \max\{i | P_i(\alpha) = 0\}$.

Proposition 3.8. *We get the following alternative descriptions:*

$$\mathscr{S}^{(k)}_{\mathscr{A}} \cong \{(C,\mu) \in \mathscr{C}_{\mathscr{A}} \times \{0,1,\dots,k-1\}^{\mathscr{A}} \,|\, \mu \text{ is } C\text{-admissible}\},$$

where $(C,\mu) \le (C',\mu')$ *iff* $\begin{cases} \mu(\alpha) \le \mu'(\alpha) \text{ for any } \alpha \in \mathscr{A}; \\ \mu(\alpha) < \mu'(\alpha) \text{ for } \alpha \text{ separating } C, C'. \end{cases}$

Corollary 3.9. *As sets this yields the following bijections:*

$$\text{for simplicial arrangements:} \quad \mathscr{S}^{(k)}_{\mathscr{A}} \cong \mathscr{C}_{\mathscr{A}} \times \{0,\dots,k-1\}^{\mathrm{rk}(\mathscr{F}_{\mathscr{A}})},$$

$$\text{for Coxeter arrangements:} \quad \mathscr{S}^{(k)}_{\mathscr{A}_W} \cong W \times \{0,\dots,k-1\}^{\mathrm{rk}(W)}.$$

Remark 3.10 (cf. B. [2]). The higher Salvetti complex $\mathscr{S}^{(k)}_{\mathscr{A}_{\mathfrak{S}_n}}$ associated to the symmetric group $\mathfrak{S}_n$ turns out to be *anti-isomorphic* to the poset of Fox-Neuwirth's [11] cell decomposition of $F(\mathbb{R}^k,n)$, and *isomorphic* to Milgram's [15] permutohedral model for $F(\mathbb{R}^k,n)$.

4. The adjacency graph

For each $\mathscr{A}$, the *adjacency graph* $\mathscr{G}_{\mathscr{A}}$ has *vertex set* $\mathscr{C}_{\mathscr{A}}$ and *edge set* $\{(C,C') \in \mathscr{C}_{\mathscr{A}} \times \mathscr{C}_{\mathscr{A}} \,|\, \exists P \in \mathscr{F}_{\mathscr{A}} : P \prec C \text{ and } P \prec C'\}$.

Since $P(\alpha) = 0$ for a unique $\alpha \in \mathscr{A}$, the edges of $\mathscr{G}_{\mathscr{A}}$ are labelled by elements of $\mathscr{A}$. Let $S(C,C') = \{\alpha \in \mathscr{A} \,|\, C(\alpha)C'(\alpha) = -1\}$ be the set of *separating "hyperplanes"*. Then:

- The edge-path of any *geodesic* joining C and C' in $\mathscr{G}_{\mathscr{A}}$ is labelled by $S(C,C')$, in particular $d(C,C') = \#S(C,C')$;
- For any $C,C',C'' : S(C,C'') = S(C,C')\Delta S(C',C'')$.

Proposition 4.1 (Björner-Edelman-Ziegler [3])**.** *The oriented matroid $\mathscr{F}_{\mathscr{A}}$ is determined by the adjacency graph $\mathscr{G}_{\mathscr{A}}$.*

This suggests the existence of a combinatorial model for $\mathscr{M}_k(\mathscr{A})$, defined directly in terms of the adjacency graph $\mathscr{G}_{\mathscr{A}}$ of $\mathscr{A}$. We present such a model for the Coxeter arrangements $\mathscr{A}_{\mathfrak{S}_n}$ and conjecturally for all Coxeter arrangements $\mathscr{A}_W$. It is a simplicial model with the remarkable property that the construction is compatible with cartesian product on the nose. (Not only up to homotopy like the higher Salvetti complexes).

Definition 4.2. *Let $E_{\mathscr{A}}$ be the simplicial set whose d-simplices are $(d+1)$-tupels $(C_0,C_1,\dots,C_d)$ of chambers. A simplicial filtration is defined by $(C_0,C_1,\dots,C_d) \in E^{(k)}_{\mathscr{A}}$ iff $(S(C_0,C_1),\dots,S(C_{d-1},C_d))$ contains $< k$ times each $\alpha \in \mathscr{A}$.*

- $E_{\mathscr{A}}$ is contractible, filtered by simplicial subsets $E^{(k)}_{\mathscr{A}}$;
- $E_{\mathscr{A}_W} = EW$ (the universal W-bundle) and $E_{\mathscr{A}_W}/W = BW$ (the classifying space for W);
- There is a filtration-preserving simplicial map $nerve(\mathscr{S}^{(k)}_{\mathscr{A}}) \to E^{(k)}_{\mathscr{A}}$ defined by $(C_0,\mu_0) \le \cdots \le (C_d,\mu_d) \mapsto (C_0,\dots,C_d)$

- $E_{\mathscr{A}\oplus\mathscr{B}} \cong E_{\mathscr{A}} \times E_{\mathscr{B}}$ compatible with filtrations.

Theorem 4.3 (Smith [22], Kashiwabara [13], B. [2]). *$|E^{(k)}_{\mathscr{A}_{\mathfrak{S}_n}}| \simeq \mathscr{M}_k(\mathscr{A}_{\mathfrak{S}_n})$. And the operad $(E^{(k)}_{\mathscr{A}_{\mathfrak{S}_n}})_{n\geq 0}$ is a simplicial E_k-operad, i.e. its geometric realization has the homotopy type of Boardman-Vogt's operad of little k-cubes.*

Conjecture 4.4 (Fiedorowicz [9]). *For any finite Coxeter group W, one has $|E^{(k)}_{\mathscr{A}_W}| \simeq \mathscr{M}_k(\mathscr{A}_W)$.*

This would extend the operad structure of the B/C/D-Coxeter groups to the higher complements of their Coxeter arrangement and yield very interesting structured simplicial E_k-operads.

REFERENCES

[1] W. Arvola – *Complexified real arrangements of hyperplanes*, Manuscripta Math. **71**(1991), 295–306.
[2] C. Berger – *Opérades cellulaires et espaces de lacets itérés*, Ann. Inst. Fourier **46**(1996), 1125–1157.
[3] A. Björner, P.H. Edelman, G.M. Ziegler – *Hyperplane arrangements with a lattice of regions*, Discr. Comp. Geom. **5**(1990), 263–288.
[4] A. Björner – *Posets, regular CW-complexes and Bruhat order*, Europ. J. Combinatorics **5** (1984), 7–16.
[5] E. Brieskorn – *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12**(1971), 57–61.
[6] P. Deligne – *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17**(1972), 273–302.
[7] A. Dimca and S. Papadima – *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements*, Ann. of Math. **158**(2003), 473–507.
[8] J. Edmonds and A. Mandel – *Topology of oriented matroids*, Notices AMS **25**(1978), A-510.
[9] Z. Fiedorowicz – *The Smith filtration for Coxeter Systems*, preprint (2002).
[10] J. Folkman and J. Lawrence – *Oriented matroids*, J. Comb. Theory, Ser. B **25**(1978), 199–236.
[11] R. Fox and L. Neuwirth – *The braid groups,* Math. Scand. **10**(1962), 119–126.
[12] I.M. Gelfand and G.L. Rybnikov – *Algebraic and topological invariants of oriented matroids*, Sov. Math. Dokl. **40**(1990), 148–152.
[13] T. Kashiwabara – *On the Homotopy Type of Configuration Complexes,* Contemp. Math. **146**(1993), 159-170.
[14] M.C. McCord – *Homotopy type comparison of a space with complexes associated with its open covers*, Proc. AMS **18**(1967), 705–708.
[15] R.J. Milgram – *Iterated loop spaces,* Ann. of Math. **84**(1966), 386-403.
[16] P. Orlik – *Complements of subspace arrangements*, J. Alg. Geom. **1**(1992), 147–156.
[17] D. Quillen – *Higher algebraic K-theory I,* Lecture Notes in Math. **341**, Springer Verlag (1973), 85-147.
[18] R. Randell – *Morse theory, Milnor fibers and minimality of hyperplane arrangements*, Proc. AMS **130**(2002), 2737–2743.
[19] C.L. Reedy – *Homotopy theories of model categories* (1973), cf. http://www.math.mit.edu/∼psh.
[20] M. Salvetti – *Topology of the complement of real hyperplanes in $\mathbb{C}^n$*, Invent. Math. **88**(1987), 603–618.
[21] M. Salvetti and S. Settepanella – *Combinatorial Morse theory and minimality of hyperplane arrangements*, Geometry and Topology **11**(2007), 1733–1766.
[22] J.H. Smith – *Simplicial Group Models for $\Omega^n S^n X$,* Israel J. of Math. **66**(1989), 330-350.

Cartier modules: finiteness results

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I report on joint work with Gebhard Böckle on the theory of Cartier modules [BB06] we are currently developing. In particular I will explain the result of our pre-print [BB09]. An important tool in the study of local cohomology modules in positive characteristic has been their natural *left* Frobenius action. This leads to deep structural results [HS93, HS77, Lyu97] which are important for the study of singularities, amongst other things.

Motivated by this success one considers a dual situation, namely that of a quasi-coherent $\mathscr{O}_X$-module M equipped with a *right* action of the Frobenius F. Just to fix notation we will always work over a scheme X over a finite field of characteristic p such that the Frobenius on X is a finite map (i.e. X is called F-finite). A right action on a $\mathscr{O}_X$-module means that M is equipped with an $\mathscr{O}_X$-linear map $C = C_M : F_*M \to M$. Such a map is called *q^{-1}-linear*, or *Cartier linear*. We call a pair (M,C) consisting of a quasi-coherent $\mathscr{O}_X$-module M and a q^{-1}-linear map C on M a *Cartier module*. The prototypical example of a Cartier module is the dualizing sheaf ω_X of a smooth and finite type k-scheme X (k perfect) together with the classical Cartier operator, [Car57, Kat70]. The relation to the *left* actions described above is via Grothendieck-Serre duality where the left action on $\mathscr{O}_X$ corresponds to the Cartier map on ω_X, *cf.* [BB06].

1. THE MAIN FINITENESS RESULT

The main structural result for Cartier modules is as follows: if X is an F-finite scheme, then

every coherent Cartier module has – up to nilpotence – finite length.

A Cartier module $M = (M,C)$ is called *nilpotent*, if $C^e = 0$ for some $e \geq 0$. This notion of nilpotence is crucial and to formalize this viewpoint one localizes the abelian category of *coherent* Cartier modules at its Serre subcategory of nilpotent Cartier modules. The localized category, which we call *Cartier crystals*, is an abelian category. Its objects are coherent Cartier modules, but the notion of isomorphism has changed: A morphism $\varphi : M \to N$ of Cartier modules induces an isomorphism of associated Cartier crystals if φ is a *nil-isomorphism*, *i.e.* both, kernel and cokernel of φ are nilpotent. Thus, more precisely, the main result can be states as:

Theorem 1.1. *If X is a locally Noetherian scheme over $\mathbb{F}_q$ such that the Frobenius F on X is a finite map, then every coherent Cartier module has – up to nilpotence – finite length. More precisely, every coherent Cartier crystal has finite length.*

Idea of the proof. Let us isolate the main points in the proof. The ascending chain condition is clear since X is Noetherian and the underlying $\mathscr{O}_X$-module of the Cartier module is coherent. Hence one has to show that any descending chain of Cartier submodules $M \supseteq M_1 \supseteq M_2 \supseteq M_3 \supseteq \ldots$ stabilizes up to nilpotence. This is shown by induction on the dimension of X. Two main ideas enter into its proof.

(1) Any coherent Cartier module (M,C) is nil-isomorphic to a Cartier submodule $(\underline{M},C)$ with surjective structural map, *i.e.* with $C(\underline{M}) = \underline{M}$. It is easy to see that the support of $\underline{M}$ is a *reduced* subscheme of X.

This part is proved by showing that the descending chain of images $M \supseteq C(M) \supseteq C^2(M) \supseteq \dots$ stabilizes and this stable image is then $\underline{M}$. This step may be viewed as a global version of an important theorem in Hartshorne and Speiser [HS77] about co-finite modules (over a complete, local, and F-finite ring) with a left action of the Frobenius. In fact, the proof in [HS77] implicitly uses Cartier modules, but is much less general than the proof we are outlining and which is due to Gabber [Gab04, Lemma 13.1] (see [KLZ07, BSTZ09] for an application of this result to the theory of test ideals, and [Bli08] for yet another version). In that same paper of Gabber one finds as well the other crucial ingredient; as part of the proof of his [Gab04, Lemma 13.3]:

(2) Let (M,C) be a coherent Cartier module with $C(M) = M$. Then there is a dense open set $U \subseteq X$ such that for all Cartier submodules $N \subseteq M$ with the same generic rank as M, the quotient M/N is supported in the closed set $Y = X \setminus U$.

The critical point is that the closed subset Y only depends on (M,C), but not on the submodule N. With 1 and 2 at hand, the proof of the above Theorem proceeds roughly as follows: By 1 we may replace the above chain by a nil-isomorphic chain where all structural maps $C_i : M_i \to M_i$ are surjective. The surjectivity is then automatically true for all quotients M/M_i. Since the generic ranks of M_i can only drop finitely many times, we can assume that – after truncating – the generic ranks are constant. Part 2 then implies that all quotients M/M_i are supported on Y. In fact, the reducedness of the support of M/M_i by 1 implies that if I is a sheaf of ideals cutting out Y, then $I \cdot (M/M_i) = 0$ and hence $IM \subseteq M_i$ for all i. So the stabilization of the chain $\{M_i\}_{i>0}$ is equivalent to the stabilization of the chain $\{M_i/M'\}_{i>0}$ where M' is the smallest Cartier submodule of M containing IM. But the latter may be viewed as a chain of Cartier modules on the lower dimensional scheme Y, hence by induction this chain stabilizes. □

There are reasons why one expects such a strong finiteness statement for Cartier modules. The most apparent one, which also explains why Gabber's [Gab04] is so extremely helpful in our proof, is as follows: Combining key results of Emerton and Kisin [EK04] or Böckle and Pink [BP09] on a Riemann-Hilbert-type correspondence in positive characteristic with our ongoing work [BB06] one expects that the category of Cartier crystals on X is equivalent to the perverse constructible sheaves (for the middle perversity) of $\mathbb{F}_q$-vectorspaces on $X_{\text{ét}}$. One of the key results in [Gab04] is that this latter category is artinian and noetherian.

2. Consequences and relation to other finiteness results.

There are some immediate formal consequences of the finite length result for Cartier crystals. Namely, one has a theory of Jordan-Hölder series, hence the notion of length of a Cartier crystal, resp. of quasi-length (*i.e.* length up to nilpotence) of a Cartier-module. This leads to version of a result of Hochster [Hoc07, (5.1) Theorem] about the finiteness of homomorphisms of Lyubeznik's F-finite modules. Our result in this context states:

Theorem 2.1. *Let X be an F-finite scheme and let $\mathscr{M},\mathscr{N}$ be Cartier crystals. Then the module* $\mathrm{Hom}_{\mathsf{Crys}}(\mathscr{M},\mathscr{N})$ *is a finite-dimensional vector space over* $\mathbb{F}_q$.

The finiteness of the homomorphism set, together with the finite length, formally implies the finiteness of submodules of any Cartier module (up to nilpotence), see also [Hoc07, (5.2) Corollary].

Theorem 2.2. *Let X be F-finite. Then a coherent Cartier module has, up to nilpotence, only finitely many Cartier submodules*

As already mentioned, the theory of $\mathscr{O}_X$-modules equipped with a left action of the Frobenius is much more extensively studied as the right actions we investigate here, and there are some deep known results which are quite similar to our Main Theorem. Examples are the above mentioned result of Hartshorne and Speiser in [HS77] about co-finite modules with a left Frobenius action, and, most importantly, Lyubeznik's result [Lyu97, Theorem 3.2] about the finite length of objects in his category of $\mathscr{F}$-finite modules over a regular ring (which is essentially of finite type over a regular local ring).

In the regular F-finite case, the connection between our and Lyubeznik's result is obtained by tensoring with the inverse of the dualizing sheaf ω_X to obtain an equivalence of Cartier modules and the category of γ-sheaves which was recently introduced in [Bli08]. γ-sheaves are $\mathscr{O}_X$-modules with a map $\gamma : M \to F^*M$, and it was shown that the associated category of γ-crystals is equivalent to the category of Lyubeznik's $\mathscr{F}$-finite modules [Lyu97] (in the affine F-finite case), resp. to Emerton and Kisin's category of locally finitely generated unit $\mathscr{O}_X[F]$-modules [EK04] (finite type over a F-finite field case). Hence our main result yields the following (partial) extension of the main result in [Lyu97, Theorem 3.2].

Theorem 2.3. *Let X be a regular F-finite scheme. Then every finitely generated unit $\mathscr{O}_X[F]$-module (resp. $\mathscr{F}$-finite module in the terminology of* [Lyu97]*) has finite length.*

The connection to Hartshorne and Speiser's theory of co-finite modules with a left Frobenius action relies on Matlis duality and has been used, at least implicitly, many times before, for an explicit use see [Gab04, BSTZ09]. What it comes down to is that if F is finite, then the Matlis dual functor commutes with F_*, and hence for a complete local and F-finite ring R this functor yields an equivalence of categories between coherent Cartier modules over $\operatorname{Spec} R$ and co-finite left $R[F]$-modules. This equivalence preserves nilpotence and hence we recover the analogous result of our main theorem for co-finite left $R[F]$-modules, *cf.* [Lyu97, Theorem 4.7].

Besides these important consequences we show how our theory implies *global* structural results which generalize analogous results for modules with a *left* Frobenius action over a *local* ring obtained recently by Enescu and Hochster [EH08], and Sharp [Sha07]. Our global version states:

Theorem 2.4. *Let X be an F-finite scheme, and (M,C) a coherent Cartier module with surjective structural map C. Then the set*

$$\{\operatorname{Supp}(M/N) | N \subseteq M \text{ a Cartier submodule}\}$$

is a finite collection of reduced sub-schemes. It consists of the finite unions of the finitely many irreducible closed sub-schemes contained in it.

In the same spirit we show the following result, which generalizes [EH08, Corollary 4.17]:

Theorem 2.5. *Let R be a Noetherian and F-finite ring. Assume that R is F-split, i.e. there is a map $C : F_* R \to R$ splitting the Frobenius.*

(1) *The Cartier module (R, C) has only finitely many Cartier submodules.*
(2) *If R is moreover complete local, then the injective hull E_R of the residue field of R has* some *left $R[F]$-module structure such that E_R has only finitely many $R[F]$-submodules.*
(3) *If $(R, \mathfrak{m})$ is also quasi-Gorenstein, then the top local cohomology module $H^d_{\mathfrak{m}}(R)$ with its standard left $R[F]$-structure has only finitely many $R[F]$-submodules.*

Note that in this result we show the finiteness of the actual number of Cartier submodules of $R[F]$-submodules, and not just the finiteness up to nilpotence!

References

[BB06] Manuel Blickle and Gebhard Böckle, *Cartier Crystals*, in preparation, started 2006.
[BB09] Manuel Blickle and Gebhard Boeckle, *Cartier modules: Finiteness results*, preprint, aug 2009.
[Bli08] Manuel Blickle, *Minimal γ–sheaves*, Algebra and Number Theory **2** (2008), no. 3, 347–368.
[BP09] Gebhard Böckle and Richard Pink, *Cohomological Theory of crystals over function fields*, to appear, 2009.
[BSTZ09] Manuel Blickle, Karl Schwede, Shunsuke Takagi, and Wenliang Zhang, *Discreteness and rationality of F-jumping numbers on singular varieties*, 29 pages, to appear in Math Ann, June 2009.
[Car57] P. Cartier, *Une nouvelle opération sur les formes différentielles*, C.R. Acad. Sci., Paris **244** (1957), 426–428.
[EH08] Florian Enescu and Melvin Hochster, *The Frobenius structure of local cohomology*, Algebra Number Theory **2** (2008), no. 7, 721–754. MR MR2460693
[EK04] Mathew Emerton and Mark Kisin, *Riemann–Hilbert correspondence for unit $\mathscr{F}$-crystals*, Astérisque **293** (2004), vi+257 pp.
[Gab04] Ofer Gabber, *Notes on some t-structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734.
[Hoc07] Melvin Hochster, *Some finiteness properties of Lyubeznik's $\mathscr{F}$-modules*, Algebra, geometry and their interactions, Contemp. Math., vol. 448, Amer. Math. Soc., Providence, RI, 2007, pp. 119–127. MR MR2389238 (2009c:13010)
[HS77] Robin Hartshorne and Robert Speiser, *Local cohomological dimension in characteristic p*, Annals of Mathematics **105** (1977), 45–79.
[HS93] Craig L. Huneke and Rodney Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 765–779.
[Kat70] Nicholas M. Katz, *Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin*, Publ. Math., Inst. Hautes Etud. Sci. **39** (1970), 175–232.
[KLZ07] Mordechai Katzman, Gennady Lyubeznik, and Wenliang Zhang, *On the discreteness and rationality of jumping coefficients*, preprint, 2007.
[Lyu97] Gennady Lyubeznik, *$\mathscr{F}$-modules: an application to local cohomology and D-modules in characteristic* $p > 0$, Journal für reine und angewandte Mathematik **491** (1997), 65–130.
[Sha07] Rodney Y. Sharp, *Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure*, Trans. Amer. Math. Soc. **359** (2007), no. 9, 4237–4258 (electronic). MR MR2309183 (2008b:13006)

The cone of Betti diagrams

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1. THE INVARIANTS OF GRADED MODULES UP TO SCALING

The basic objects under study are the finitely generated graded modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$ over a standard graded polynomial ring $S = k[x_1,x_2,\dots,x_n] = \bigoplus_{i\geq 0} S_i$ over a field k.

The basic invariants of these modules are the *Hilbert function*, $H_M(i) = \dim_k M_i$ and the *graded Betti numbers*, $\beta_{i,j}(M) = \dim_k \operatorname{Tor}_i^S(M,k)_j$. The latter are computed from a *minimal free resolution*

$$0 \longleftarrow M \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_p \longleftarrow 0$$

where the free modules can be chosen to be graded as $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}(M)}$ and the maps to be homogeneous of degree zero. The array of Betti numbers is usually referred to as a *Betti diagram* och *Betti table*. The Betti numbers are finer invariants than the Hilbert function, and we have the relation

$$\operatorname{Hilb}(M,t) = \sum_{j\in\mathbb{Z}} H_M(j)t^j = \frac{1}{(1-t)^n}\sum_{i=0}^{p}\sum_{j\in\mathbb{Z}}(-1)^i\beta_{i,j}(M)t^j.$$

The characterization problem asks: *What values can these invariants take?*

In the case of the Hilbert function of cyclic modules $M = S/I$, this question was answered by Macaulay [13] by giving the necessary and sufficient inequalities:

$$0 \leq H_M(i+1) \leq H_M(i)^{\langle i\rangle}, \qquad i = 0,1,2,\dots,$$

where $a^{\langle i\rangle}$ is a combinatorially defined function.

For the graded Betti numbers, much less has been known. In 1994 Bigatti [1], Hulett [12] and Pardue [14] proved that fixing the Hilbert function, the Betti numbers are bounded by the Betti numbers of the lexicographic module. These Betti numbers can be obtained from the Eliahou-Kervaire [8] formula. Furthermore, Peeva [15] proved in 2004 that any Betti diagram can be obtained from the maximal Betti diagram with the same Hilbert function by a sequence of *adjacent cancellations*, i.e. by decreasing $\beta_{i,j}$ and $\beta_{i+1,j}$ by one simultaneously. However, which cancellations lead to actual Betti diagram is highly unknown.

While working with J. Söderberg on the Multiplicity Conjecture by Herzog, Huneke and Srinivasan [11], we came up with a new idea for studying the characterization problem for Betti numbers. Instead of asking what all possible Betti diagrams of graded modules are we asked the following:

> *What are the possible Betti diagrams of graded modules up to positive rational multiples? That is, what is the cone that the Betti diagrams span?*

One way to see that this question might have a more reasonable anwser is to look at what happens with the Hilbert functions under the same assumptions. We get that Macaulay's bound up to scaling is transformed into the following simple linear inequalities:

$$\frac{H_M(0)}{H_S(0)} \geq \frac{H_M(1)}{H_S(1)} \geq \cdots \geq \frac{H_M(i)}{H_S(i)} \geq \frac{H_M(i+1)}{H_S(i+1)} \geq \cdots$$

We conjectured the answer for the question about the cone of Betti diagrams to be the following:

Conjecture 1 (B.- Söderberg [4]). *The cone of Betti diagrams is spanned by the Cohen-Macaulay pure diagrams and the natural partial ordering defines a structure of a simplicial fan on this cone.*

Herzog and Kühl [10] had shown that for a Cohen-Macaulay module with a *pure resolution*, the Betti numbers are uniquely determined up to a constant factor as

$$\beta_{i,d_i} = \lambda(-1)^i \prod_{j \neq i} \frac{1}{d_j - d_i}$$

where $d_0, d_1, \ldots, d_p$ are the only degrees where the Betti numbers are non-zero for the various homological degrees. Hence the conjecture gave a very precise description of the cone of Betti diagrams. The combinatorial structure comes from the fact that in the natural partial ordering given by $\mathbf{d} \leq \mathbf{d}'$ if $d_i \leq d_i'$, for all i, every maximal chain of pure Cohen-Macaulay Betti diagrams spans a simplicial cone of maximal dimension and these cones fit together nicely in a simplicial fan.

Part of this conjecture was proven by Eisenbud, Fløystad and Weyman in 2007 using Schur modules:

Theorem 1 (Eisenbud-Fløystad-Weyman [5]). *Assume that* $\mathrm{char}\, k = 0$. *Then, for any degree sequence* $d_0, d_1, \ldots, d_n$*, there is an artinian module with pure resolution and* $\beta_{i,j}(M) = 0$ *for* $j \neq d_i$, $i = 0, 1, \ldots, n$.

Relatively soon after that, in the beginning of 2008, Eisenbud and Schreyer proved the remaining parts of our conjecture.

Theorem 2 (Eisenbud-Schreyer [6]). *Any Betti diagram of a Cohen-Macaulay graded module can be uniquely written as a positive sum of pure Cohen-Macaulay diagrams.*

Moreover, the corresponding statement is true for cohomology tables of vector bundles on projective spaces.

The proof of the conjecture and the fascinating correspondence between Betti diagrams and cohomology tables heavily rely on the combinatorial description of the simplicial fan which allowed Eisenbud and Schreyer to write down the inequalities given by the supporting hyperplanes of the facets of the cone. From these inequalities, they recognized patterns coming from cohomology tables of certain vector bundles and the conjecture was proven by establishing a positivity result for a pairing of Betti diagrams and cohomology tables.

Later on, we extended the statement on the Betti diagram side to modules that are not necessarily Cohen-Macaulay [3] and Eisenbud and Schreyer extended the statement on the cohomology table side to coherent sheaves [7], now allowing infinite sums of cohomology tables.

2. Multiple components by studying the cone

When we have a description of the cone of Betti diagrams, we can use this to find instances where there are several components in the parameter space of modules with a given Hilbert funtion. In order to get a nice parameter space, like in the case of the Hilbert scheme, we need to look at modules with their presentations, as we look at subschemes with their embeddings. In this way, we get the parameter space

$$\mathrm{GradMod}_F(H)$$

parametrizing quotients $F \longrightarrow M \longrightarrow 0$ with a given Hilbert function H.

Fixing the Hilbert function, we intersect the cone of Betti diagrams with an affine space and thus we get a convex set in this affine space of potential Betti diagrams.

It is known that the Betti numbers are upper semi-continuous and thus we get that the generic point of any component of $\mathrm{GradMod}_F(H)$ must have minimal Betti numbers among all the modules in the same component (cf. Ragusá and Zappala [16]).

When studying the cone of Betti diagrams, we see that there is oftentimes not a unique minimal element in the convex set given by the Hilbert function.

Example 1. Looking at the case of a codimension three Hilbert function $H = (3,9,6,4)$, we have four possible minimal Betti diagrams in *Macaulay2* notation [9] :

$$\begin{array}{r|cccc} & 3 & 12 & 16 & 7 \\ \hline 0: & 3 & . & . & . \\ 1: & . & 12 & 10 & 3 \\ 2: & . & . & . & . \\ 3: & . & . & 6 & 4 \end{array} \qquad \begin{array}{r|cccc} & 3 & 13 & 16 & 6 \\ \hline 0: & 3 & . & . & . \\ 1: & . & 12 & 10 & 2 \\ 2: & . & . & . & . \\ 3: & . & 1 & 6 & 4 \end{array} \qquad \begin{array}{r|cccc} & 3 & 14 & 16 & 5 \\ \hline 0: & 3 & . & . & . \\ 1: & . & 12 & 10 & 1 \\ 2: & . & . & . & . \\ 3: & . & 2 & 6 & 4 \end{array} \qquad \begin{array}{r|cccc} & 3 & 15 & 16 & 4 \\ \hline 0: & 3 & . & . & . \\ 1: & . & 12 & 10 & . \\ 2: & . & . & . & . \\ 3: & . & 3 & 6 & 4 \end{array}$$

In each of these cases, we can find a module with this Betti diagram as a direct sum of other modules, for example, we have that

$$\begin{array}{r|cccc} & 3 & 12 & 16 & 7 \\ \hline 0: & 3 & . & . & . \\ 1: & . & 12 & 10 & 3 \\ 2: & . & . & . & . \\ 3: & . & . & 6 & 4 \end{array} = \begin{array}{r|cccc} & 1 & 6 & 8 & 3 \\ \hline 0: & 1 & . & . & . \\ 1: & . & 6 & 8 & 3 \\ 2: & . & . & . & . \\ 3: & . & . & . & . \end{array} + \begin{array}{r|cccc} & 2 & 6 & 8 & 4 \\ \hline 0: & 2 & . & . & . \\ 1: & . & 6 & 2 & . \\ 2: & . & . & . & . \\ 3: & . & . & 6 & 4 \end{array}$$

3. What happens for multiples of a given Hilbert function?

Proposition 1. *If M corresponds to a smooth point of* $\mathrm{GradMod}_F(H)$*, then M^n corresponds to a smooth point of* $\mathrm{GradMod}_{F^n}(nH)$ *and the dimension of the component containing M^n is n^2 times the dimension of the component containing M.*

Example 2. When multiplying the Hilbert function in Example 1 by $n = 2,3,4$, we get more and more possible minimal Betti diagrams. We can find constructions for all of them, and in all cases we get that the difference between the dimension of the tangent space and obstruction space is $32n^2$.

In the cases described above, the component structure gets more complicated as we take multiples of the Hilbert function. The following example shows that this is not always the case.

Example 3. In an article with Iarrobino [2] we proved that in the case of $H = (1,3,4,4)$, the parameter space of artinian algebras having Hilbert function H has two components, both of dimension 8, corresponding to the two mininal Betti diagrams:

	1	6	9	4
0 :	1	.	.	.
1 :	.	2	.	.
2 :	.	.	1	.
3 :	.	4	8	4

and

	1	6	9	4
0 :	1	.	.	.
1 :	.	2	1	.
2 :	.	1	.	.
3 :	.	3	8	4

However, when we multiply the Hilbert function by two, $\mathrm{GradMod}_{S^2}(2H)$ has a single component of dimension 32 with the minimal Betti diagram

	1	10	16	8
0 :	2	.	.	.
1 :	.	4	.	.
2 :	.	.	.	.
3 :	.	6	16	8

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References

[1] A. M. Bigatti. Upper bounds for the Betti numbers of a given Hilbert function. *Comm. Algebra*, 21(7):2317–2334, 1993.

[2] M. Boij and A. Iarrobino. Reducible family of height three level algebras. *J. Algebra*, 321:86–104, 2009.

[3] M. Boij and J. Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case, 2008. Preprint, arXiv.org:0803.1645.

[4] M. Boij and J. Söderberg. Graded Betti numbers of Cohen-Macaulay modules and the multiplicity conjecture. *J. Lond. Math. Soc. (2)*, 78(1):85–106, 2008.

[5] D. Eisenbud, G. Fløystad, and J. Weyman. The existence of pure free resolutions. Preprint, arXiv.org: 0709.1529.

[6] D. Eisenbud and F.-O. Schreyer. Betti numbers of graded modules and cohomology of vector bundles. *J. of Amer. Math. Soc.*, 22:859–888, 2009.

[7] D. Eisenbud and F.-O. Schreyer. Cohomology of coherent sheaves and series of supernatural bundles, 2009.

[8] S. Eliahou and M. Kervaire. Minimal resolutions of some monomial ideals. *J. Algebra*, 129(1):1–25, 1990.

[9] D. R. Grayson and M. E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[10] J. Herzog and M. Kühl. On the Betti numbers of finite pure and linear resolutions. *Comm. Algebra*, 12(13-14):1627–1646, 1984.

[11] J. Herzog and H. Srinivasan. Bounds for multiplicities. *Trans. Amer. Math. Soc.*, 350(7):2879–2902, 1998.

[12] H. A. Hulett. Maximum Betti numbers of homogeneous ideals with a given Hilbert function. *Comm. Algebra*, 21(7):2335–2350, 1993.

[13] F. S. Macaulay. Some properties of enumeration in the theory of modular systems. *Proc. London Math. Soc.*, 26:531–555, 1927.

[14] K. Pardue. Deformation classes of graded modules and maximal Betti numbers. *Illinois J. Math.*, 40(4):564–585, 1996.

[15] I. Peeva. Consecutive cancellations in Betti numbers. *Proc. Amer. Math. Soc.*, 132(12):3503–3507 (electronic), 2004.

[16] A. Ragusa and G. Zappalá. On the reducibility of the postulation Hilbert scheme. *Rend. Circ. Mat. Palermo (2)*, 53(3):401–406, 2004.

Smash Induction

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Given two spaces X and Y, there is a Künneth formula for the homology of the product $X \times Y$. The aim of this work is to extend this formula to the context of smash induction of spectra. The motivation for such an extension comes from topological Hochschild homology, where smash-inductions with respect to cyclic groups are used to construct an approximation, called topological cyclic homology, to algebraic K-theory.

The spectral sequence for smash induction can be thought of as a three-step generalization of the Künneth formula.

Firstly, there is a Tor or Künneth spectral sequence for the smash product $E \wedge F$ of two spectra, where the Tor groups are taken over the graded ring $\pi_* \mathbb{S}$ of homotopy groups of the sphere spectrum. More generally, if the spectra E and F are module spectra over some ring spectrum R, there is a *TorBrun* spectral sequence of the form

$$E^2_{s,t} = \mathrm{Tor}^{\pi_* R}(\pi_* E, \pi_* F) \Rightarrow \pi_*(E \wedge_R F)$$

converging towards the homotopy groups of the smash-product over R. If $R = Hk$ is the Eilenberg–MacLane spectrum of a field and $E = R \wedge \Sigma^\infty X_+$ and $F = R \wedge \Sigma^\infty Y_+$ are the spectra computing homology of X and Y with coefficients in k, then the Tor-spectral sequence is concentrated on a line and the Künneth formula can be read off from it.

Secondly, there is a Tor spectral sequence, due to Lewis and Mandell [1], for smash-products of spectral with group-actions. This spectral sequence is most easily described by considering stable homotopy groups as Mackey-functors. More precisely, if a finite group G acts on the spectrum E, then the assignment taking a subgroup H to the set of G-homotopy classes of G-maps from the suspension spectrum $\Sigma^\infty G/H_+$ to E forms a Mackey functor. This is the zeroth homotopy group of E considered as a Mackey functor. The other homotopy groups of E can be considered as Mackey functors in a similar way. Lewis and Mandell have proved that there is a Tor-spectral sequence for spectra with group action which is completely analogous to the Tor spectral sequence for spectra without group-actions. The only difference is that the spectral sequence for spectra with G-action is a spectral sequence of Mackey functors.

The underlying reason for the Tor spectral sequence is that for sectra of the form $E = R \wedge \Sigma^\infty S^m$ and $F = R \wedge \Sigma^\infty S^n$, the homotopy of the smash product $E \wedge_R F$ is isomorphic to the tensor product of the $\pi_* R$-modules $\pi_* E$ and $\pi_* F$.

Let E and F be module spectra over a ring spectrum R with action of G. The underlying reason for the Tor-spectral sequence for G-spectra is that if E and F are of the form $E = R \wedge \Sigma^\infty (S^m \wedge G/H_+)$ and $F = R \wedge \Sigma^\infty (S^n \wedge G/K_+)$, then the graded Mackey functor $\pi_*(E \wedge_R F)$ is isomorphic to the box product $\pi_* E \,\square_{\pi_* R}\, \pi_* F$ of graded Mackey functors.

The third step from smash products to smash induction involves a construction of S. Bouc called tensor induction [2]. It will be convenient to write $S_G F$ for the smash induction $\wedge_{g \in G} F$

with G acting by permutation of smash factors. If R is a ring spectrum, then $S_G R$ is a ring spectrum with G-action. Suppose that F is a left R-module so that $S_G F$ is a left $S_G R$-module and let E be a right $S_G R$-module. If E and F are free in the sense that $E = S_G R \wedge \Sigma^\infty (S^m \wedge G/H_+)$ and $F = R \wedge \Sigma^\infty S^n$, then there is an isomorphism of graded Mackey functors of the form

$$\pi_n(E \wedge_{S_G R} S_G F) \cong \bigoplus_{s+t=n} T^{\pi_* R}_{s,t}(\pi_* E, \pi_* F),$$

where $T^{\pi_* R}_{*,*}(\pi_* E, \pi_* F)$ is a slight variation on Bouc's tensor induction of Mackey functors. The spectral sequence for smash induction takes the form

$$E^2_{s,t} = T^{\pi_* R}_{s,t}(\pi_* E, \pi_* F) \Rightarrow \pi_{s+t}(E \wedge_{S_G R} S_G F)$$

for E a general right $S_G R$ module spectrum and F a general left R module spectrum.

REFERENCES

[1] L. G. Lewis and M. A. Mandell Equivariant universal coefficient and Künneth spectral sequences. *Proc. London Math. Soc. (3)*, 92:505–544, 2006.

[2] S. Bouc Non-additive exact functors and tensor induction for Mackey functors. *Mem. Amer. Math. Soc.*, 144, 2000.

Koszul cycles and the syzygies of Veronese algebras

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This is a report on a joint work with Tim Römer and Winfried Bruns.

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring over a field and $R = S/I = \bigoplus_{i=0}^{\infty} R_i$ a graded quotient of it. Let $\mathfrak{m}_R$ denote the maximal homogeneous ideal of R. For every $c \in \mathbb{N}$ we consider the c-th Veronese algebra $R^{(c)}$ of R which is defined as:

$$R^{(c)} = \bigoplus_{i=0}^{\infty} R_{ic}.$$

The question we address is how the degrees of the syzygies of $R^{(c)}$ vary with c. Here we consider $R^{(c)}$ as a standard graded algebra (i.e. generated over K by its component of normlized degree 1) have a surjective K-algebra map $T \to R^{(c)}$ where T is the symmetric algebra of the vector space R_c. So T is itself a polynomial ring over K whose Krull dimension equals the vector space dimension of R_c. We want to understand the degrees of the syzygies of $R^{(c)}$ as a T-module.

Many invariants are used to measure the degrees of the syzygies. We recall those that play a role in our discussion. Given a graded and finitely generated R-module M we denote the (i,j)-th Betti number of M as an R-module by $\beta_{ij}^R(M)$. We set $t_i^R(M) = \sup\{j : \beta_{ij}^R(M) \neq 0\}$,

$$\operatorname{reg}_R(M) = \sup\{t_i^R(M) - i : i \geq 0\},$$

$$\operatorname{slope}_R(M) = \sup\{\frac{t_i^R(M) - t_0^R(M)}{i} : i > 0\}$$

and $\operatorname{Rate}(R) = \operatorname{slope}_R(\mathfrak{m}_R)$. While $\operatorname{reg}_R(M)$ can be infinite, $\operatorname{slope}_R(M)$ is finite for every finitely generated graded R-module M, see [2]. By definition, R is Koszul if and only if $\operatorname{Rate}(R) = 1$. The Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ of M is, by definition, $\operatorname{reg}_S(M)$; it is finite and does not depend on S. The Green-Lazarsfeld index of R, denoted by $\operatorname{index}(R)$, is defined as:

$$\operatorname{index}(R) = \sup\{p : t_i^S(R) \leq i+1 \text{ for every } i \leq p\}.$$

The following facts about the syzygies of the Veronese algebras are classical and well-known:

(1) $S^{(c)}$ is defined by quadrics, i.e. $\operatorname{index}(S^{(c)}) \geq 1$,
(2) The Castelnuovo-Mumford regularity $\operatorname{reg}(S^{(c)})$ of $S^{(c)}$ equal to $n - \lceil n/c \rceil$,
(3) if R is defined by equations of degree a and smaller then $R^{(c)}$ is defined by equations of degree $\leq \max(2, \lceil a/c \rceil)$.

Let $K(\mathfrak{m}_R^c, R)$ denote the Koszul complex over R associated to the c-th power of $\mathfrak{m}_R$, by $H_\bullet(\mathfrak{m}_R^c, R)$ its homology, by $Z_\bullet(\mathfrak{m}_R^c, R)$ its cycles and by $B_\bullet(\mathfrak{m}_R^c, R)$ its boundaries. One notices that

$$\beta_{ij}^T(R^{(c)}) = \dim_K H_i(\mathfrak{m}_R^c, R)_{jc}.$$

Hence the study of the syzygies of $R^{(c)}$ is essentially equivalent to the study of the Koszul homology modules $H_i(\mathfrak{m}_R^c, R)$. Taking into account that $\mathfrak{m}_R^c H_i(\mathfrak{m}_R^c, R) = 0$ one can set up an inductive procedure leading to bounds for the regularity of $Z_\bullet(\mathfrak{m}_R^c, R)$. It follows that:

Theorem 1. $H_i(\mathfrak{m}_R^c, R)_j = 0$ for $j \geq (i+1)c + \min(i\operatorname{Rate}(R), i + \operatorname{reg}(R))$. In particular, we have $\operatorname{index}(R^{(c)}) \geq c - \operatorname{reg}(R)$ and $\operatorname{index}(R^{(c)}) \geq c$ if R is Koszul.

Theorem 1 has been proved by Green for Veronese subrings $S^{(c)}$ of polynomial rings S in characteristic 0, see [4]. For $n = 2$ or $c = 2$ the ring $S^{(c)}$ has a determinantal presentation and (at least in characteristic 0) the value of $\operatorname{index}(S^{(c)})$ can be deduced from the known resolutions of it. One has:

$$\operatorname{index}(S^{(c)}) = \begin{cases} \infty & \text{if } n = 2 \text{ or } (n = 3 \text{ and } c = 2) \\ 5 & \text{if } n > 3 \text{ and } c = 2. \end{cases}$$

Note however that Andersen [1] has showed that $\operatorname{index}(S^{(2)}) = 4$ in characteristic 5 if $n \geq 7$. In characteristic 0 and for $n > 2$ and $c > 2$ Ottaviani and Paoletti [5] have proved that

$$\operatorname{index}(S^{(c)}) \leq 3c - 3$$

with equality if $n = 3$. They conjectured that equality holds for every $n \geq 3$. We prove that the bound and the equality for $n = 3$ hold independently of the characteristic. In the proof an important role is played by the duality:

$$\dim_K H_i(\mathfrak{m}_S^c, S)_j = \dim_K H_{N-n-i}(\mathfrak{m}_S^c, S)_{Nc-n-j}$$

where $N = \dim S_c$ and the fact that for $n = 3$ the regularity of $S^{(c)}$ is ≤ 2. The duality above can be seen as a special instance of a duality of Avramov-Golod type, which is the algebraic counterpart of Serre duality. Our main contribution to the problem of finding $\operatorname{index}(S^{(c)})$ is the following improvement of Green's lower bound:

Theorem 2. If K has characteristic 0 or $> c+1$ then $\operatorname{index}(S^{(c)}) \geq c+1$ for every n.

For $c = 3$ and K of characteristic 0 Theorem 2 has been proved by Rubei in [6]. Set $Z_t = Z_t(\mathfrak{m}_S^c, S)$ and $B_t = B_t(\mathfrak{m}_S^c, S)$ and let Z_1^t denote the image of $\wedge^t Z_1$ in Z_t. The proof of Theorem 2 is based on three facts:

1) Z_t / Z_1^t is generated in degree $< (c+1)i$,

2) for every $a \in \mathbb{N}$ with $1 \geq a < c$, and for polynomials $f_1, \dots, f_{t+1} \in S_a$ and $g_1, \dots, g_{t+1} \in S_{c-a}$ one has

$$\sum_{\sigma \in \mathbb{S}_{t+1}} (-1)^\sigma f_{\sigma(t+1)} \otimes f_{\sigma(1)} g_1 \wedge f_{\sigma(2)} g_2 \wedge \cdots \wedge f_{\sigma(t)} g_t \in Z_t$$

where $\mathbb{S}_{t+1}$ is the symmetric group.

3) $(c+1)!\mathfrak{m}_S^{c-1} Z_1^c \subset B_c$

Indeed, the cycles in 2) are used together with a symmetrization argument to prove 3). Combining 1) and 3) one shows that $H_i(\mathfrak{m}_S^c)_{ic+j} = 0$ for $j \geq i + c - 1$ and $i \geq c$ which, in turn, implies that $\operatorname{index}(S^{(c)}) \geq c+1$. The cycles described in 2) can be "explained" in terms of multilinear algebra and diagonal maps between symmetric and exterior powers of vector spaces. There are some indications that those cycles might generate $Z(\mathfrak{m}_S^c, S)$ as an S-algebra. For general R we prove the following:

Theorem 3. One has $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ for every $c \geq \operatorname{slope}_S(R)$.

As shown in [2], $\operatorname{slope}_S(R) = 2$ if R is Koszul. In particular, if R is Koszul then $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$ for every $c \geq 2$.

References

[1] J. L. Andersen, *Determinantal rings associated with symmetric matrices: a counterexample*, Ph.D. Thesis, University of Minnesota (1992).

[2] L. L. Avramov, A. Conca, S. B. Iyengar, *Free resolutions over commutative Koszul algebras*, arXiv:0904.2843, (2009).

[3] W. Bruns, A. Conca, T. Römer, *Koszul homology and syzygies of Veronese subalgebras*, arXiv:0902.2431 (2009).

[4] M. Green, *Koszul cohomology and the geometry of projective varieties II*, J. Differ. Geom. **20**, 279–289, (1984).

[5] G. Ottaviani and R. Paoletti, *Syzygies of Veronese embeddings*, Compos. Math. **125**, 31–37, (2001).

[6] E. Rubei, *A result on resolutions of Veronese embeddings*, Ann. Univ. Ferrara, Nuova Ser., Sez. VII **50**, 151–165 (2004).

Parametrizations of Ideals in $K[x,y]$ and $K[x,y,z]$

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Let K be a filed of any characteristic and $K[x_1,\ldots,x_n]$ the polynomial ring in n variables. For a polynomial $f \in K[x_1,\ldots,x_n]$ and any term order τ we denote by $\mathrm{in}_\tau(f)$ the initial term of f with respect to the term order τ. If $I \subset K[x_1,\ldots,x_n]$ is an ideal, we denote by $\mathrm{in}_\tau(I)$ the initial ideal of I with respect to τ, that is the monomial ideal generated by $\mathrm{in}_\tau(f)$, for all $f \in I \setminus \{0\}$.
Given a monomial ideal $I_0 \subset K[x_1,\ldots,x_n]$, the set

$$V_h(I_0) = \{I \subset K[x_1,\ldots,x_n] \ : \ I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety, in the sense that an ideal $I \in V(I_0)$ can be considered as a point in the affine space $\mathbb{A}^N$. The coordinates are given by the coefficients of the non-leading terms in the reduced Gröbner basis of the ideal I. If $\dim_K(K[x_1,\ldots,x_n]/I_0) < \infty$, also the set in which we consider all ideals (not necessarily homogeneous), $V(I_0) := \{I \subset K[x_1,\ldots,x_n] \ : \ \mathrm{in}(I) = I_0\}$ has a structure of affine variety.

These varieties play an important role in the study of various types of Hilbert schemes and also in the problem of deforming nonradical to radical ideals, see [4, 8, 10, 11, 13, 14, 15, 16].

The main goal of this paper is to parametrize the affine variety $V(I_0)$, when I_0 is a monomial ideal of $R = K[x,y]$, τ is the degree reverse-lexicographic (DRL) term order induced by $x > y$, and $\dim_k(R/I_0) < \infty$. In the above situation it is known by results of J. Briançon [4] and A. Iarrobino [15] that $V(I_0)$ is an affine space. This fact is also a consequence of general results of A. Bialynicki-Birula [1, 2].

The parametrization is obtained in the following way. Let $I_0 \subset R$ be a monomial ideal with $\dim_k(R/I_0) < \infty$. We choose for I_0 the following set of generators:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}),$$

where $t := \mathrm{Min}\{j : x^j \in I_0\}$, $m_0 = 0$ and $m_i := \mathrm{Min}\{j : x^{t-i}y^j \in I_0\}$ for every $1 \le i \le t$. Notice that we have $0 = m_0 \le m_1 \le \ldots \le m_t$; so let us define $d_i := m_i - m_{i-1}$ for all $i = 1,\ldots t$. Consider the following matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

This matrix is a Hilbert-Burch matrix for I_0, in the sense that its signed minors are $x^{t-i}y^{m_i}$, so they generate the ideal, and its columns generate their syzygy module. Consider the corresponding degree matrix $U(I_0) = (u_{i,j})$, where the entries of $U(I_0)$ are the degrees of the (homogeneous) entries representing a map of degree 0 from $F_1 = \bigoplus_{i=1}^{t} R(-t+i-m_i-1)$ to $F_0 = \bigoplus_{i=0}^{t} R(-t+i-1-m_{i-1})$. So we have:

$$u_{i,j} = i-j+m_j-m_{i-1}, \text{ for } i=1,\dots,t+1 \text{ and } j=1,\dots,t.$$

Now let A be another $(t+1)\times t$ matrix, with entries in the polynomial ring in one variable $K[y]$, with the property:

$$\deg(a_{i,j}) \le \begin{cases} \operatorname{Min}\{u_{i,j}-1, \quad d_i-1\} & \text{if } i \le j, \\ \operatorname{Min}\{u_{i,j} \quad\quad , \quad d_j-1\} & \text{if } i > j, \end{cases} \tag{0.1}$$

where $i=1,\dots,t+1$ and $j=1,\dots,t$. We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition. Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$, where N is the sum over i and j of the above bounds $+1$, whenever these bounds are positive.

Now we define $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ to be the application given by

$$\psi(A) := I_t(X+A),$$

where by $I_t(X+A)$ is the ideal generated by t-minors of the matrix $X+A$. The parametrization that we are looking for is given by the following theorem:

Theorem 1. *Let $I_0 \subset R = K[x,y]$ be a monomial lex-segment ideal with $\dim(R/I_0) < \infty$. Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.*

This parametrization allows us to find a formula for the dimension of this affine space in terms of the Hilbert function of R/I_0. For every $i \ge 0$ we denote by $h_i = \dim_K((R/I_0)_i)$, that is the value of the Hilbert function of R/I_0 in i.

Proposition 2. *Let $I_0 \subset R$ be a monomial lex-segment ideal, then*

$$\dim(V(I_0)) = \dim_K(R/I_0) + 1 + \sum_{i\ge 1} h_i(h_{i-1}-h_{i-2}).$$

We will see that $\mathscr{A}_{I_0}$ also parametrizes an affine open subset of the Hilbert function strata of $Hilb^n(\mathbb{P}^2)$, when the characteristic of the filed is 0. This means that the formula found in Proposition 2 is also valid for $Hilb^H(\mathbb{P}^2)$, where H denotes the Hilbert function of an ideal $I \subset K[x,y,z]$ defining a zero-dimensional scheme in $\mathbb{P}^2$. The dimension of $Hilb^H(\mathbb{P}^2)$ was determined by G.Gotzmann in [12]. A different formula was also given by G. Ellingsrud and S. A. Strømme in [10, 11], then by A. Iarrobino and V. Kanev in [17]. The latest formula was found by K. De Naeghel and M. Van den Bergh in [19]. Our formula is nearest to the latter one.

Due to the nature of the proof of Proposition 2 we have the following corollary.

Corollary 3. *Let H and h be as above and let K be a field of characteristic 0. Denote by $\operatorname{Lex}(h)$ the lex-segment ideal of $R = K[x,y]$ with Hilbert function h, by $\mu = \mu(\operatorname{Lex}(h))$ the number of its minimal generators and by $n = \dim_K(R/\operatorname{Lex}(h))$. For $n \ge 2$ we have*

$$\operatorname{Max}\{n+\mu-1,\ n+2\} \ \le\ \dim Hilb^H(\mathbb{P}^2) \ \le\ 2n.$$

The lower bound generalizes slightly the result obtained by K. De Naeghel and M. Van den Bergh in [19, Corollary 6.2.3].

Given any monomial ideal J_0 of $S = K[x,y,z]$ and considering the affine variety of the homogeneous ideals that have I_0 as initial ideal for a certain term order τ, we do not obtain in general the affine space (see [3] and [9] for examples). However, if we take $J_0 = I_0S$, with $I_0 \in K[x,y]$ a lex-segment ideal, and choose the degree reverse-lexicographic order induced by $x > y > z$, then $V_h(J_0)$ is again the affine space. We find a parametrization for this space, which comes from the parametrization of $V(I_0)$ in the following way.

Definition 4. Let $A \in \mathscr{A}_{I_0}$, with entries $a_{i,j}$. For every $i = 1,\dots,t+1$ and $j = 1,\dots,t$ we define:

$$a_{i,j}^{\underline{hom}} := z^{u_{i,j}-\deg(a_{i,j})} a_{i,j}^{hom},$$

where $a_{i,j}^{hom}$ is the standard homogenization. The homogenization of the matrix A will be the matrix with entries $a_{i,j}^{\underline{hom}}$. We will denote this matrix by $A^{\underline{hom}}$.

Then we define the following application:

$$\overline{\psi} : \mathscr{A}_{I_0} \longrightarrow \overline{V}(J_0)$$
$$\overline{\psi}(A) = I_t(X + A^{\underline{hom}}), \quad \text{for all } A \in \mathscr{A}_{I_0}.$$

With the above notation and some technical remarks that we skip here we obtain:

Theorem 5. *Let $J_0 = I_0S \subset S$ be a monomial ideal, where I_0 is a lex-segment ideal of R such that $\dim_K(R/I_0) < \infty$. Then the application $\overline{\psi} : \mathscr{A}_{I_0} \longrightarrow \overline{V}(J_0)$ defined above is a bijection.*

This extension will allow us to study the Betti strata of $V(J_0)$, with $J_0 \subset S$ as above. In [16, Remark 3.7] A. Iarrobino gives a generalization to codimension two punctual schemes in $\mathbb{P}^2$ of the codimension formula of the Betti strata, together with an indication of a proof. We obtain a different proof for the above-mentioned formula. We show that $V(J_0)$ is dense in $\mathbb{G}(H)$, where H is the Hilbert series of S/J_0 and

$$\mathbb{G}(H) = \{J \subset S \ : \ J \text{ is a 0-dimensional scheme } H_{S/J} = H\}$$

is the variety that parametrizes graded homogeneous ideals of S with Hilbert series H. In this case H will be of the form

$$H(s) = \frac{h(s)}{1-s},$$

with $h(s)$ the Hilbert series of the zero-dimensional algebra $S/J + (\ell)$, where ℓ is a linear non-zero divisor of S/J.

From now on we will assume that the field K is algebraically closed and its characteristic is either 0 or large enough with respect to the degree of $h(s)$. We will skip the proof of the following crucial remark:

Remark 6. The set

$$\mathbb{G}^*_{Lex}(H) = \{J \in \mathbb{G}^*(H) \ : \ \mathrm{in}(J) = \mathrm{Lex}(h)S\}$$

is an open subset of $\mathbb{G}(H)$.

For a homogeneous ideal $J \subset S$ we will denote by $\beta_{i,j}(J)$ the (i,j)th Betti number. In particular, $\beta_{0,j}(J)$ is the number of minimal generators of J of degree j. It is known that any two of the sets $\{\beta_{0,j}(J)\}_j$, $\{\beta_{1,j}(J)\}_j$ and $\{\dim(J_j)\}_j$ determine the third. For the fixed Hilbert series $H(s) = h(s)/(1-s)$ and for given integers j and u we define:

$$\begin{aligned} V(H,j,u) &= \{J \in \mathbb{G}^*_{Lex}(H) \,:\, \beta_{0,j}(J) = u\}, \\ V(H,j,\geq u) &= \{J \in \mathbb{G}^*_{Lex}(H) \,:\, \beta_{0,j}(J) \geq u\}. \end{aligned}$$

For a vector $\beta = (\beta_1, \dots, \beta_j, \dots)$ with integral entries we define:

$$\begin{aligned} V(H,\beta) &= \textstyle\bigcap_j \; V(H,j,\beta_j), \\ V(H,\geq\beta) &= \textstyle\bigcap_j \; V(H,j,\geq\beta_j). \end{aligned}$$

Finally, skipping all the technical details, we present the following proposition and its two corollaries.

Proposition 7. *Each $V(H,j,\geq\beta_j)$ is a determinantal variety and the variety $V(H,\geq\beta)$ is the transversal intersection of the $V(H,j,\geq\beta_j)$'s. The variety $V(H,j,\geq\beta_j)$ is irreducible and it coincides with the closure of $V(H,j,\beta_j)$, provided $V(H,j,\beta_j)$ is not empty.*

Corollary 8.
1. *The variety $V(H,\geq\beta)$ is irreducible.*
2. *The codimension of $V(H,\geq\beta)$ is the sum of the codimensions of the $V(H,j,\geq\beta_j)$'s.*

Corollary 9. *Let $J \in \mathbb{G}(H)$ and set $\beta = (\beta_{0,j}(J))$. The variety $V(H,\geq\beta)$ is irreducible, it is the closure of $V(H,\beta)$ and it has codimension*

$$\sum_j \beta_{1,j}(J)\beta_{0,j}(J).$$

References

[1] A. Bialynicki-Birula, *Some Theorems on Actions of Algebraic Groups*, Annals of Mathematics Vol.**98**, No. **3**, (1973), 480–497.

[2] A. Bialynicki-Birula, *Some Properties of the Decompositions of Algebraic Varieties Determined by Actions of a Torus*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences math. astr. et phys. **24**(**9**), (1976), 667–674.

[3] M. Boij, A. Iarrobino, *Reducible Family of Height Three Level Algebra*, to appear after revision, J. Algebra.

[4] J. Briançon, *Description de $Hilb^n\mathbb{C}\{x,y\}$*, Invent. Math. **41**, (1977), 45–89.

[5] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, No. **39**, revised edition (1998), Cambridge, U.K.

[6] W. Bruns, U. Vetter, *Determinantal Rings*, Springer Lecture Notes in Mathematics **1327**, (1988)

[7] CoCoATeam, *CoCoA: a system for doing Computations in Commutative Algebra*, available at http:// cocoa.dima.unige.it

[8] A. Conca, M.E. Rossi, G.Valla, *Gröbner Flags and Gorenstein Algebras*, Compositio Math. **129**, no. 1, 95–121, (2001).

[9] A. Conca, G. Valla, *Canonical Hilbert-Burch Matrices for Ideals of $K[x,y]$*. Michigan Math. J. **57**, 157–172, (2008).

[10] G.Ellingsrud, S.A.Strømme, *On the Homology of the Hilbert Scheme of Points in the Plane*, Invent. Math. **87**, 343–352, (1987).

[11] G.Ellingsrud, S.A.Strømme, *On a Cell Decomposition of the Hilbert Scheme of Points in the Plane*, Invent. Math. **91**, 365–370, (1988).

[12] G. Gotzmann *A Stratification of the Hilbert Scheme of Points in the Projective Plane*, Math. Z. **199**, no.4, 539–547, (1988).

[13] L.Göttsche, *Betti-Numbers of the Hilbert Scheme of Points on a Smooth Projective Surface*, Math. Ann. **286**, 193–207, (1990).

[14] L.Göttsche, *Betti-numbers for the Hilbert Function Strata of the Punctual Hilbert Scheme in two Variables*, Manuscripta Math. **66**, 253–259, (1990).

[15] A. Iarrobino, *Punctual Hilbert Schemes*, Mem. Amer. Math. Soc. 10 (188), (1977).

[16] A. Iarrobino, *Betti Strata of Height Two Ideals*. Journal of Algebra 285. 835-855. (2005)

[17] A. Iarrobino, V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lecture Notes in Mathematics, **1721**, Springer-Verlag, Berlin, (1999).

[18] M. Kreuzer, L. Robbiano, *Computational Commutative Algebra 1,2*, Springer, (2000, 2005).

[19] K. De Naeghel, M. Van den Bergh, *Ideal classes of three dimensional Artin-Schelter regular algebras*, J. Algebra **283** , no. 1, 399–429, (2005).

[20] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, American Mathematica Society, University Lecture Series, Volume 8, (1995).

Gröbner bases of ideals cogenerated by Pfaffians

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This is a report on a joint work with Enrico Sbarra.

Let $X = (X_{ij})$ be a skew-symmetric $n \times n$ matrix of indeterminates and let $K[X] := K[X_{ij}\, 1 \leq i < j \leq n]$ be the polynomial ring over the field K. Denoted by $[a_1, \ldots, a_{2t}]$ the Pfaffian of the skew-symmetric submatrix of X with row and column indexes $a_1 < \ldots < a_{2t}$. We say that the *size* of α is $2t$. Let us recall the definition of partial order on the set $P(X)$ of all Pfaffians of X, as introduced in [DP].

Let $\alpha = [a_1, \ldots, a_{2t}], \beta = [b_1, \ldots, b_{2s}] \in P(X)$. Then

$$\alpha \leq \beta \quad \text{if and only if} \quad t \geq s \text{ and } a_i \leq b_i \text{ for } i = 1, \ldots, 2s.$$

Definition 1. *The ideal cogenerated by $\alpha \in P(X)$ is the ideal of $K[X]$:*

$$I_\alpha(X) := (\beta \in P(X)\, \beta \not\geq \alpha).$$

The well studied ideal generated by all Pfaffians of size $2r$ of X, is nothing but the ideal cogenerated by $\alpha = [1, \ldots, 2r-2]$.

Recall that a *standard monomial* is a product $\alpha_1 \cdot \ldots \cdot \alpha_h$ of Pfaffians with $\alpha_1 \leq \ldots \leq \alpha_h$. Since $K[X]$ is an ASL on $P(X)$ (see [DP]), standard monomials form a basis of $K[X]$ as a K-vector space and, since $I_\alpha(X)$ is an order ideal, one has:

Proposition 2. *Standard monomials $\alpha_1 \cdot \ldots \cdot \alpha_h$ with $\alpha_1 \leq \ldots \leq \alpha_h$ and $\alpha_1 \not\geq \alpha$ form a K-basis of $I_\alpha(X)$.*

We recall that a term order is said to be anti-diagonal if the initial monomial of the Pfaffian $[a_1, ..., a_{2t}]$ is its main anti-diagonal, i.e.

$$\mathrm{in}([a_1, ..., a_{2t}]) = X_{a_1 a_{2t}} X_{a_2 a_{2t-1}} \cdot \ldots \cdot X_{a_t a_{t+1}}.$$

By using the structure of ASL and the original KRS correspondence introduced in [K], Herzog and Trung proved that the natural generators of $I_{2r}(X)$ form a Gröbner basis with respect to any anti-diagonal term order [HT]. In a subsequent remark they ask whether their result can be extended to any cogenerated Pfaffian ideal. This question is very natural and in the analogue cases of ideals of minors of a generic matrix and of a symmetric matrix the answer is affirmative, as proved respectively in [HT] and [C].

Quite surprisingly in this case the answer is negative:

Example 3. *Let X be a 6×6 skew symmetric matrix of indeterminates, and let $I_\alpha(X)$ be the Pfaffian ideal cogenerated by $\alpha = [1,2,4,5]$. The natural generators of $I_\alpha(X)$ are $[1,2,3,4,5,6]$, $[1,2,3,4]$, $[1,2,3,5]$, $[1,2,3,6]$. The element $[1,2,3,4][1,5] - [1,2,3,5][1,4]$ belongs to $I_\alpha(X)$ but its initial term is not divisible by any of the leading terms of the generators.*

By using a modified version of the KRS correspondence introduced in [Bu], we characterize what we call *G-Pfaffians* , i.e. the Pfaffians α with the property that the natural generators of $I_\alpha(X)$ are a G-basis.

Theorem 4. *The natural generators of $I_\alpha(X)$ form a G-basis of $I_\alpha(X)$ w.r.t. any antidiagonal term order if and only if $\alpha = [a_1,...,a_{2t}]$, with $a_i = a_{i-1}+1$ for $i = 3,\dots,2t-1$.*

From now on we assume that α is a G-Pfaffian. It is easy to see that the ideals cogenerated by $[a_1,\dots,a_{2t}]$ and by $[1,a_2-a_1+1,\dots,a_{2t}-a_1+1]$ have the same numerical invariants, thus we may assume that

$$\alpha = [1,a,a+1,\dots,a+2t-3,b].$$

Since the initial ideal of $I_\alpha(X)$ is square free, we consider the associated simplicial complex Δ_α, which is the family of all subsets Z of $X_+ := \{(i,j)|1 \le i < j \le n\}$ such that $\Pi_{(i,j)\in Z} X_{ij} \notin \mathrm{in}(I_\alpha(X))$.

To describe the faces of Δ_α we decompose any set $Z \subset X_+$ into disjoint chains according to a "light and shadow" procedure. We use two lights, one coming from the lower-left side and one coming from the upper-right side. We let

$$\delta(Z) := \{(i,j) \in Z \ \nexists (i',j') \in Z \text{ with } i' > i, j' < j\},$$

$$\delta'(Z) := \{(i,j) \in Z \ \nexists (i',j') \in Z \text{ with } i' < i, j' > j\}.$$

We characterize the faces of Δ_α by decomposing $Z = Z_1 \cup Z_2 \cup \ldots \cup Z_r$, where $Z_1 = \delta'(Z)$, $Z_2 := \delta(Z \setminus Z_1)$ and $Z_h := \delta(Z \setminus \cup_{k<h} Z_k)$, and by giving conditions on the various components of the decomposition, see [DeS, Prop. 3.5]. In the following theorem we describe the facets of Δ_α. Denote a saturated chain of X_+ with starting point Q and ending point P by QP, and a union of non-intersecting saturated chains by $\sqcup$.

Theorem 5. *Let $Q = (1,a)$, $Q_i = (a,a+2i-1)$ for $i = 1,\dots,t-2$ and $P_j = (n-2j+1,n)$ for $j = 1,\dots,t$. Furthermore, let $Q_{t-1} = (a,k)$, $Q^h = (h,b)$, $P_{hk} = (h,k)$, with $h,k \in \mathbb{N}$. Then*

$$Z \text{ is a facet of } \Delta_\alpha \quad \text{iff} \quad Z = (QP_{hk} \sqcup Q^h P_t) \sqcup_{i=1}^{t-1} Q_i P_i,$$

for some $h \in \{1,\dots,a-1\}$ and $k \in \{a+2t-3,\dots,b-1\}$.

In Figure 1 an example of facet is given. As a consequence one has:

Corollary 6. *Let α be a G-Pfaffian. Then Δ_α is pure and shellable, in particular it is Cohen-Macaulay.*

Moreover the multiplicity of $K[X]/I_\alpha(X)$ is equal to the number of the facets of Δ_α. For the sake of notational simplicity, we now modify slightly the notation introduced in Theorem 5 and we let $Q_t = (h,b)$. Furthermore, let $A_{hk} = (a^{hk}_{ij})$ be the $t \times t$ matrix with entries

$$a^{hk}_{ij} = \binom{x_{P_j} + y_{P_j} - x_{Q_i} - y_{Q_i}}{x_{P_j} - x_{Q_i}} - \binom{x_{P_j} + y_{P_j} - x_{Q_i} - y_{Q_i}}{x_{P_j} - y_{Q_i}},$$

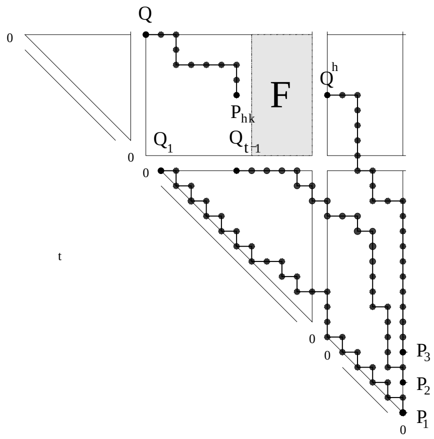

Fig.1: a facet in the case $t = 3$

where x_p and y_p denote the coordinates of a point P in X_+. By using a result of [GV] we prove:

$$e(K[X]/I_\alpha(X)) = \sum_{\substack{h=1,\dots,a-1\\ k=a+2t-3,\dots,b-1}} \binom{h+k-a-1}{h-1} \det A_{hk}.$$

References

[Bu] W. Burge. "Four Correspondences Between Graphs and Generalized Young Tableaux", *J. Combin. Theory* (A) **17** (1974), 12-30.

[C] A. Conca. "Gröbner bases of ideals of minors of a symmetric matrix", *J. Algebra* **166**-2 (1994), 406-421.

[DP] C. De Concini, C. Procesi. "A characteristic free approach to invariant theory", *Adv. in Math.* **21** (1976), 330-354.

[DeS] E. De Negri, E. Sbarra. " "Gröbner bases of ideals cogenerated by Pfaffians", preprint 2009.

[GV] I. M. Gessel, G. Viennot. "Binomial determinants, paths, and hook length formulae", *Adv. Math.* **58** (1985), 300-321.

[HT] J. Herzog, N. V. Trung. "Gröbner bases and multiplicity of determinantal and Pfaffian ideals", *Adv. Math.* **96** (1992), 1-37.

[K] D. Knuth. "Permutations, matrices, and generalized Young tableaux", *Pacific J. Math.* **34** (1970), 709-727.

Symmetric powers of vector bundles and Hilbert-Kunz theory

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Let R be a normal, d-dimensional standard-graded ring of characteristic $p > 0$. Let $I = (f_1, \dots, f_N) \subseteq R$ be a homogeneous R_+-primary ideal. In Hilbert-Kunz theory, we consider the colengths of the Frobenius powers

$$I^{[q]} = (f_1^q, \dots, f_N^q)$$

for varying p-powers q. In particular, we have an invariant $e_{HK}(I)$, the *Hilbert-Kunz multiplicity* of I, which describes the asymptotic behavior of these colengths:

$$e_{HK}(I) := \lim_{e \to \infty} \frac{\operatorname{length} R/I^{[p^e]}}{p^{de}}.$$

Monsky has shown that this limit always exists and that it is a positive real number. Now one may ask:

- Is the Hilbert-Kunz multiplicity always a rational number?
- In a relative situation over Spec $\mathbb{Z}$, does the limit for $p \to \infty$ exist?

For two-dimensional rings ($d = 2$), Brenner and Trivedi have answered both questions affirmatively. In higher dimensions the answers are still unknown. In the two-dimensional case, the key idea is that all the information we need to describe $e_{HK}(I)$ is encoded in the *Harder-Narasimhan filtration* ("HNF") of the syzygy bundle $\operatorname{Syz}(f_1, \dots, f_N)$ associated to I on $\operatorname{Proj} R$.

Theorem 1 (Harder-Narasimhan)**.** In arbitrary characteristic and dimension, any vector bundle $\mathscr{E}$ on $Y := \operatorname{Proj} R$ has a unique filtration such that the quotients are semistable of decreasing slopes.

Now let $\mathscr{E} = \operatorname{Syz}_1(f_1, \dots, f_N)$, $d_k := \deg f_k$, and let r_i, μ_i denote the ranks and slopes in the HNF of $\mathscr{E}$.

Theorem 2 (Brenner, Trivedi)**.** For $d = 2$ in the relative situation over $\mathbb{Z}$, we have

$$\lim_{p \to \infty} e_{HK}(I_p) = \frac{1}{2}\Big(-\frac{1}{\delta}\sum_{i=1}^{t} r_i \mu_i^2 - \delta \sum_{k=1}^{N} d_k^2\Big) =: e_{HK}^0(I),$$

where $\delta = \deg Y = \deg \mathscr{O}_Y$ is the degree of the projective curve $Y = \operatorname{Proj} R$.

Next we define an invariant $\tau(\operatorname{Syz}_1)$ which is defined in any characteristic and has a description in terms of the *Hilbert-Kunz slope* $\mu_{HK}(\operatorname{Syz}_1) := \sum_{i=1}^{t} r_i \mu_i^2$, which yields a connection to the Hilbert-Kunz multiplicity via the above formula.

Let R be a normal, standard-graded, d-dimensional algebra over a field k (any characteristic), and let $\mathscr{E}$ be a vector bundle on $Y = \operatorname{Proj} R$.

Definition 3. The *n-th divided power* of $\mathscr{E}$ is

$$D^n(\mathscr{E}) := S^n(\mathscr{E}^\vee)^\vee$$

(canonically isomorphic to the n-th symmetric power $S^n(\mathscr{E})$ in characteristic 0). Let β be a variable taking values in $\mathbb{Z}$, and let

$$\tau(\beta, \mathscr{E}) := \lim_{n\to\infty} \frac{\sum_{-\beta n}^{\beta n} h^0(D^n(\mathscr{E})(m))}{n^d \cdot \operatorname{rk} D^n(\mathscr{E})}.$$

If this limit exists and is a polynomial in β, let $\tau(\mathscr{E})$ be the constant term in this polynomial.

Theorem 4. For $d = 2$ and characteristic 0 we have:

(a)
$$\tau(\mathscr{E}) = \frac{1}{\delta r(r+1)} \sum_{i=1}^{t} \binom{r_i+1}{2} \mu_i^2 + \sum_{i<j} r_i r_j \mu_i \mu_j,$$
where r_i, μ_i are the ranks and slopes in the HNF and $r = \operatorname{rk}\mathscr{E}$.

(b)
$$\tau(\mathscr{E}) = \frac{1}{2r(r+1)}(\mu_{HK}(\mathscr{E}) + (\deg \mathscr{E})^2).$$

(c) Let $I = (f_1, \ldots, f_N)$ be a homogeneous, primary ideal, $d_k = \deg f_k$, then
$$e^0_{HK}(I) = N(N-1)\tau(\operatorname{Syz}_1) - \frac{\delta}{2}((\sum_{k=1}^{N} d_k)^2 + \sum_{k=1}^{N} d_k^2).$$

The theorem shows that in ring dimension two, the τ invariant is indeed related to the Hilbert-Kunz multiplicity. Using Koszul complexes, we can compute the τ invariant numerically.

Let I be an ideal as in the theorem, and consider the presenting sequence for its first syzygy bundle:

$$0 \longrightarrow \operatorname{Syz}_1 \longrightarrow \bigoplus_{k=1}^N \mathscr{O}_Y(-d_k) \xrightarrow{f_1,\ldots,f_N} \mathscr{O}_Y \longrightarrow 0\,.$$

Now we consider the associated Koszul complex, which is a short exact sequence in this case:

$$0 \longrightarrow D^n(\operatorname{Syz}_1) \longrightarrow D^n(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k))$$
$$\longrightarrow D^{n-1}(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k)) \otimes \bigwedge^1 \mathscr{O}_Y = D^{n-1}(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k))$$
$$\longrightarrow D^{n-2}(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k)) \otimes \bigwedge^2(\mathscr{O}_Y) = 0.$$

Twisting by $m \in \mathbb{N}_0$ and taking sheaf cohomology we obtain an exact sequence

$$0 \longrightarrow H^0(Y, D^n(\operatorname{Syz}_1)) \longrightarrow H^0(Y, D^n(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k)))$$
$$\xrightarrow{\psi_{n,m}} H^0(Y, D^{n-1}(\bigoplus_{k=1}^N \mathscr{O}_Y(-d_k))) \longrightarrow \operatorname{coker} \longrightarrow 0\,.$$

The map $\psi_{n,m}$ is readily available in sparse matrix form, so we can compute its kernel for each m and n and hence we can numerically compute the τ invariant, for example using the computer algebra package Macaulay2. If we consider the cokernel of $\psi_{n,m}$ instead of the kernel, we obtain a new invariant $\sigma(I)$ which turns out to be even better suited for our needs and can be computed using Macaulay2 or CoCoA.

Definition 5.

$$\sigma(I) := \lim_{n\to\infty} \frac{\sum_{m=0}^{\infty} \dim_k \operatorname{coker} \psi_{n,m}}{\text{normalizing term}}$$

where the normalizing term ensures that the limit exists; for $d = 2$ we use the normalizing term $\binom{n+N-1}{N}$.

We come to our main result.

Theorem 6. For $d = 2$ and characteristic 0, we have $\sigma(I) = e^0_{HK}(I)$. In particular, in the relative situation we have

$$\lim_{p\to\infty} e_{HK}(I_p) = \sigma(I).$$

It should be possible to generalize this result to the case of positive characteristic, $d = 2$. Our main theorems will still hold provided that the HNF of the syzygy bundle is "strong". We conjecture that $\sigma(I) = e_{HK}(I)$ holds in general. This is equivalent to saying that the relation between $\tau(\mathscr{E})$ and $\mu_{HK}(\mathscr{E})$ holds for $\mathscr{E} = \mathrm{Syz}_1$ if $\mu_{HK}(\mathscr{E})$ is defined in terms of the so-called *strong HNF*.

In higher dimensions however, it is not clear how a suitable invariant should even be defined. Only some "trial definitions" exist so far, for example: $\sigma(I)$; $\tau(\mathrm{Syz}_1)$; a variant of $\sigma(I)$ with "correcting terms" related to the Schur modules $S^{\lambda}(\mathrm{Syz}_1)$; and $\tau(\mathrm{Syz}_2^{\vee})$.

Convex normality of lattice polytopes with long edges

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All our polytopes are assumed to be convex. For a polytope P the set of its vertices will be denoted by $\mathrm{vert}(P)$.

A polytope $P \subset \mathbb{R}^d$ is *lattice* if $\mathrm{vert}(P) \subset \mathbb{Z}^d$, and P is *rational* if $\mathrm{vert}(P) \subset \mathbb{Q}^d$.

Let $P \subset \mathbb{R}^d$ be a lattice polytope and denote by L the subgroup of $\mathbb{Z}^d$, affinely generated by the lattice points in P; i. e., $L = \sum_{x,y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x-y) \subset \mathbb{Z}^d$. (Observe, $P \cap L = P \cap \mathbb{Z}^d$.)

Definition 1. ([7, Def. 2.59]) Let $P \subset \mathbb{R}^d$ be a lattice polytope.

(i) P is *integrally closed* if the following condition is satisfied:

$$c \in \mathbb{N},\ x \in cP \cap \mathbb{Z}^d \quad \Longrightarrow \quad \exists x_1, \dots, x_c \in P \cap \mathbb{Z}^d \quad x_1 + \cdots + x_c = x.$$

(ii) P is *normal* if the following condition is satisfied:

$$c \in \mathbb{N},\ x \in cP \cap L \quad \Longrightarrow \quad \exists x_1, \dots, x_c \in P \cap L \quad x_1 + \cdots + x_c = x.$$

The normality property is invariant under isomorphisms of lattice polytopes, and the property of being integrally closed is invariant under an affine change of coordinates, leaving the lattice structure $\mathbb{Z}^d \subset \mathbb{R}^d$ invariant. Here two lattice polytopes $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^{d'}$ are said to be *isomorphic* if there is an affine map (i. e., a map respecting barycentric coordinates) $\phi : \mathbb{R}^d \to \mathbb{R}^{d'}$ which restricts to a bijection $\phi : P \cap \mathbb{Z}^d \to Q \cap \mathbb{Z}^{d'}$.

It is easily observed that a lattice polytope $P \subset \mathbb{R}^d$ is integrally closed if and only if it is normal and L is a direct summand of $\mathbb{Z}^d$. Obvious examples of normal but not integrally closed polytopes are *empty* lattice simplices of large volume. No classification of empty simplice is known in all dimensions ≥ 4. The difficulty is the lack of a satisfactory characterization of the lattice widths of empty lattice simplices; see [12, 18] and the many references therein.

On the other hand, a normal polytope $P \subset \mathbb{R}^d$ can be made into a full-dimensional integrally closed polytope by changing the lattice of reference $\mathbb{Z}^d$ to L and the ambient Euclidean space $\mathbb{R}^d$ to the subspace $\mathbb{R}L$. In particular, normal and integrally closed polytopes refer to same isomorphism classes of lattice polytopes. In the combinatorial/toric geometry literature the difference between 'normal' and 'integrally closed' is often blurred.

Normal/integrally closed polytopes enjoy popularity in algebraic combinatorics. They have been showcased on recent workshops ([1, 2]). These polytopes represent the homogeneous case of the Hilbert bases of finite positive rational cones and the connection to algebraic geometry is that normal polytopes define projectively normal embeddings of toric varieties. There are many challenges of number theoretic, ring theoretic, homological, and K-theoretic nature, concerning the associated objects: Ehrhart series', rational cones, toric rings, and toric varieties; see, for example, [7].

It is easily observed that if a lattice polytope is covered by (and, especially, subdivided into) integrally closed polytopes than it is itself integrally closed. On the other hand, the simplest integrally closed polytopes one can think of are the *unimodular simplices*, i. e., the lattice simplices $\Delta = \operatorname{conv}(x_1,\dots,x_k) \subset \mathbb{R}^d$, $\dim\Delta = k-1$, for which the set $\{x_1-x_j,\dots,x_{j-1}-x_j,x_{j+1}-x_j,\dots,x_k-x_j\}$ is a part of a basis of $\mathbb{Z}^d$ for some (equivalently, every) index $j\in\{1,\dots,k\}$.

However, not all 4-dimensional integrally closed polytopes are triangulated into unimodular simplices [9, Prop. 1.2.4], and not all 5-dimensional integrally closed polytopes are covered by unimodular simplices [5] – contrary to what had been conjectured before [17]. Further 'negative' results, such as [4] and [8] (disproving a conjecture from [10]), contributed to the current thinking in the area that there is no succinct characterization of the normality property in terms of the geometry of the polytope. One could even conjecture that in higher dimensions the situation gets as bad as it can; see the discussion at the end of [2, p. 2313].

'Positive' results can be obtained while trying to understand what special constructions of lattice polytopes lead to normal, unimodularly triangulated, or unimodularly covered polytopes. Knudsen-Mumford's classical theorem ([13, Chap. III], [7, Sect. 3B]) says that every lattice polytope P has a multiple cP for some $c\in\mathbb{N}$ that is triangulated into unimodular simplices. The existence of a dimensionally uniform lower bound for the factors c seems to be a very hard problem. More recently, it was shown in [6] that there exists a dimensionally uniform exponential lower bound for unimodularly covered multiple polytopes. By improving one crucial step in [6], von Thaden was able cut down the bound to a degree 6 polynomial function of the dimension [7, Sect. 3C], [19].

The mentioned results on unimodular covers and triangulations of multiple polytopes yield no new examples of normal polytopes, though. In fact, an easy argument ensures that for any lattice polytope P the multiples cP, $c\geq \dim P-1$, are integrally closed [9, Prop. 1.3.3], [11]. However, that argument does not allow a modification that would apply to lattice polytopes with long edges of independent lengths.

For polytopes, arising in a different context and admitting unimodular triangulations as a certificate of normality, see [3, 14, 15, 16]

The question whether all smooth polytopes are normal *(Oda's question)* has attracted much attention recently. A lattice polytope $P\subset\mathbb{R}^d$ is called *smooth* if the primitive edge vectors at every vertex of P define a part of a basis of $\mathbb{Z}^d$. Such polytopes define smooth projective toric varieties. This question, together with higher syzygy analogues of the associated toric ring, such as quadratic generation or the Koszul property, was exactly the focus of [1, 2]. The fact that so far no smooth polytope just without a unimodular triangulation has been found illustrates how limited our understanding in the area is.

Motivated by Oda's question, it has been proposed that

Conjecture 2. ([2, p. 2310]) (a) Simple lattice polytopes with 'long' edges are normal, where 'long' means some invariant uniform in the dimension;

(b) Let P be a simple lattice polytope. Let k be the maximum over the heights of Hilbert basis elements of tangent cones to vertices of P. Then, if any edge of P has length $\geq k$, the polytope P is normal.

Here: (i) ‘long’ is understood in the sense of the lattice length, i. e., the number of lattice points on the edge minus 1, (ii) ‘tangent cone’ means the cone spanned by P at the given vertex, also known as the *corner cone*, and (iii) ‘height’ is understood in the sense of some positive $\mathbb{Z}_+$−grading of the affine monoid of lattice points in the tangent cone. Recall, a polytope P is called *simple* if every vertex of P is adjacent to $\dim P$ edges. See [7, Chap. 1, 2, 4] for the terminology used above.

Observe that Conjecture 2(b) implies the normality of smooth polytopes because smooth polytopes are simple with $k = 1$. It is not clear what the relationship between the two parts of Conjecture 2 is – one can easily show that there is no dimensionally uniform upper bound for the heights of Hilbert basis elements.[1]

Below we introduce the notion of convex normality that is stronger than the property of being integrally closed. It involves rational polytopes which is necessary as the proof of the main result can *not* be carried out on the level of integral polytopes. Admittedly, unimodular simplices are *not* convex-normal.

Definition 3. A rational polytope $P \subset \mathbb{R}^d$ is said to be *convex-normal* if the following condition is satisfied

$$cP = \bigcup_{\substack{v\in \mathrm{vert}(P) \\ x\in (c-1)P\bigcap\left((c-1)v+\mathbb{Z}^d\right)}} x+P, \qquad c\in\mathbb{Q}_{\geq 1}. \tag{0.1}$$

Lemma 4. (a) If P is lattice and convex-normality then for every natural number c we have

$$cP = \bigcup_{x\,\in\,\mathbb{Z}^d\bigcap(c-1)P} x+P.$$

(b) All lattice polytopes that are covered by lattice parallelotopes are convex-normal.

(c) All lattice convex-normal polytopes are integrally closed.

Define the *lattice length* of a rational segment $[x,y] \subset \mathbb{R}^d$ as the ratio of its Euclidean length and that of the primitive (i. e., with coprime components) integer vector in the direction of $y-x$. The following theorem proves Conjecture 2(a) in a strong form:

Theorem 5. (a) If every edge of a (not necessarily simple) rational polytope P has lattice length $\geq 2(\dim P)^2 + 2\dim P$ then P is convex-normal. In particular, if every edge of a lattice polytope P contains at least $\geq 2(\dim P)^2 + 2\dim P + 1$ lattice points then P is integrally closed.

(b) If P is a rational simplex whose every edge has lattice length $(\dim P)^2 + \dim P$ then P is covered by lattice parallelotopes. In particular, P is integrally closed.

Remark 6. (a) One can strengthen Theorem 5(a) in such a way that, assymtotically as $d \to \infty$, the estimate $\geq 2(\dim P)^2 + 2\dim P$ changes to $(\dim P)^2 + \dim P$. But the price for the improvement is an overcomplicated statement and proof of the main result.

[1] In another interpretation of Conjecture 2(b), the (not necessarily integral) heights of Hilbert basis elements of a simplicial cone are measured with respect to the extremal generators of the cone; in this version of the conjecture, one has (b) $\Longrightarrow$ (a).

(b) Theorem 5 is proved in two steps. First one shows that a neighborhood of the boundary $\partial(cP) \subset cP$ inside of certain width is contained in the r.h.s. of the equality (0.1). This is done by induction on dimension. The second step is to propagate lattice 'corner parallelotopal covers' from a vertex of cP deep inside cP so that only the neighborhood of the same width of the opposite part of $\partial(cP)$ may be left uncovered. This step does not need induction.

What makes the strategy work is that, in gauging the mentioned neighborhoods in two different steps, the same factor $(\dim P)^2 + \dim P$ shows up for essentially unrelated reasons. This may be a hint that any substantial improvement of the estimate in Theorem 5 (to, say, a linear function of $\dim P$) will require an entirely new approach.

REFERENCES

[1] Workshop: Combinatorial Challenges in Toric Varieties. The workshop held April 27 to May 1, 2009, Organized by Joseph Gubeladze, Christian Haase, and Diane Maclagan, American Institute of Mathematics, Palo Alto. http://www.aimath.org/pastworkshops/toricvarieties.html.

[2] Mini-Workshop: Projective Normality of Smooth Toric Varieties. *Oberwolfach Rep.*, 4(3):2283–2319, 2007. Abstracts from the mini-workshop held August 12–18, 2007, Organized by Christian Haase, Takayuki Hibi and Diane Maclagan, Oberwolfach Reports. Vol. 4, no. 3.

[3] Matthias Beck and Serkan Hoşten. Cyclotomic polytopes and growth series of cyclotomic lattices. *Math. Res. Lett.*, 13:607–622, 2006.

[4] Winfried Bruns. On the integral Carathéodory property. *Experiment. Math.*, 16:359–365, 2007.

[5] Winfried Bruns and Joseph Gubeladze. Normality and covering properties of affine semigroups. *J. Reine Angew. Math.*, 510:161–178, 1999.

[6] Winfried Bruns and Joseph Gubeladze. Unimodular covers of multiples of polytopes. *Doc. Math.*, 7:463–480, 2002.

[7] Winfried Bruns and Joseph Gubeladze. *Polytopes, rings, and K-theory*. Springer Monographs in Mathematics. Springer, Dordrecht, 2009.

[8] Winfried Bruns, Joseph Gubeladze, Martin Henk, Alexander Martin, and Robert Weismantel. A counterexample to an integer analogue of Carathéodory's theorem. *J. Reine Angew. Math.*, 510:179–185, 1999.

[9] Winfried Bruns, Joseph Gubeladze, and Ngô Viêt Trung. Normal polytopes, triangulations, and Koszul algebras. *J. Reine Angew. Math.*, 485:123–160, 1997.

[10] W. Cook, J. Fonlupt, and A. Schrijver. An integer analogue of Carathéodory's theorem. *J. Combin. Theory Ser. B*, 40:63–70, 1986.

[11] Günter Ewald and Uwe Wessels. On the ampleness of invertible sheaves in complete projective toric varieties. *Results Math.*, 19:275–278, 1991.

[12] Christian Haase and Günter M. Ziegler. On the maximal width of empty lattice simplices. *European J. Combin.*, 21:111–119, 2000. Combinatorics of polytopes.

[13] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I.* Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin, 1973.

[14] Tomonori Kitamura. Gröbner bases associated with positive roots and Catalan numbers. *Osaka J. Math.*, 42(2):421–433, 2005.

[15] Hidefumi Ohsugi and Takayuki Hibi. Unimodular triangulations and coverings of configurations arising from root systems. *J. Algebraic Combin.*, 14(3):199–219, 2001.

[16] Victor Reiner and Volkmar Welker. On the Charney-Davis and Neggers-Stanley conjectures. *J. Combin. Theory Ser. A*, 109(2):247–280, 2005.

[17] Andras Sebö. Hilbert bases, Carathéodory's theorem and combinatorial optimization. *Proc. of the IPCO conference (Waterloo, Canada)*, pages 431–455, 1990.

[18] András Sebő. An introduction to empty lattice simplices. In *Integer programming and combinatorial optimization (Graz, 1999)*, volume 1610 of *Lecture Notes in Comput. Sci.*, pages 400–414. Springer, Berlin, 1999.

[19] Michael von Thaden. Improved bounds for covers and triangulations of lattice polytopes. Preprint.

A combinatorial model for derived categories of regular toric schemes

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A toric scheme $X = X_\Sigma$ over a commutative ring A comes equipped with a preferred covering by open affine sets. From a combinatorial point of view X is specified by a finite fan Σ in $\mathbb{Z}^n \otimes \mathbb{R} \cong \mathbb{R}^n$, and each cone $\sigma \in \Sigma$ corresponds to an A-algebra A^σ and hence to an open affine set $U_\sigma = \mathrm{Spec}(A^\sigma) \subseteq X$. By evaluating on the open sets U_σ we see that a chain complex Y of quasi-coherent sheaves on X_Σ can thus be specified by a collection of A^σ-module chain complexes Y^σ for $\sigma \in \Sigma$, together with restriction maps (restricting sections to smaller open sets).

We are thus led to introduce the category $\mathbf{Pre}(\Sigma)$ of (twisted) diagrams

$$\Sigma^{\mathrm{op}} \to \text{chain complexes}, \ \sigma \mapsto Y^\sigma$$

where Y^σ is a chain complex of A^σ-modules, and the structure map corresponding to an inclusion of cones $\tau \subseteq \sigma$ is A^σ-linear. As an example, the structure sheaf gives rise to the obvious diagram $\sigma \mapsto A^\sigma$.

Not every diagram in $\mathbf{Pre}(\Sigma)$ comes from a chain complex of quasi-coherent sheaves; those which do are precisely the diagrams satisfying an additional compatibility condition, namely that for each inclusion $\tau \subseteq \sigma$ we have an isomorphism of chain complexes

$$A^\tau \otimes_{A^\sigma} Y^\sigma \cong Y^\tau \ . \tag{0.1}$$

In the language of sheaves, this means that we recover Y^τ by restricting the sections Y^σ over U_σ to the smaller open set U_τ. We denote the full subcategory of $\mathbf{Pre}(\Sigma)$ formed by the chain complexes of quasi-coherent sheaves by $\mathfrak{Qco}(X_\Sigma)$. This latter category admits a "model structure" with weak equivalences the objectwise quasi-isomorphisms. The associated homotopy (or derived) category, which is obtained by formally inverting the weak equivalences, is $D(\mathfrak{Qco}(X_\Sigma))$, the classical (unbounded) derived category of quasi-coherent sheaves on X_Σ.

It is sometimes desirable to describe the derived category of X_Σ using $\mathbf{Pre}(\Sigma)$ instead since, due to the absence of the gluing condition (0.1), this category is far easier to handle than $\mathfrak{Qco}(X)$. Now $\mathbf{Pre}(\Sigma)$ admits a model structure with weak equivalences the objectwise quasi-isomorphisms as well. However, the associated homotopy (or derived) category is *not* equivalent to $D(\mathfrak{Qco}(X_\Sigma))$: it turns out that inverting objectwise quasi-isomorphism is not enough.

One thus has to pass to a larger class of morphisms to invert instead. Here the combinatorial theory of toric schemes comes into the picture again. Each cone $\mu \in \Sigma$ determines a support function χ of Σ, that is, a function that is linear on each cone, and takes integral values on the lattice $\mathbb{Z}^n$. Such a function determines elements χ_σ such that for $\tau \subseteq \sigma$ the quotient χ_σ/χ_τ is a unit in A^τ, and hence determines a line bundle (a quasi-coherent sheaf that is locally free of rank 1) denoted $\mathcal{O}(\mu)$ by twisting the inclusion map $A^\sigma \to A^\tau$ by the quotient of χ_σ/χ_τ. (The support function associated to the cone μ is characterised by taking the value 1 on each primitive generator of μ, and taking the value 0 on all other primitive generators of cones in Σ.)

Given a diagram $Y \in \mathbf{Pre}(\Sigma)$ and a cone $\mu \in \Sigma$ we define the μth twist of Y, denoted $Y(\mu)$, by objectwise tensor product (over the appropriate rings) of the entries of Y and $\mathscr{O}(\mu)$. We now call a map $f\colon X \to Y$ in $\mathbf{Pre}(\Sigma)$ a *colocal equivalence* if the map $\hom(\mathscr{O}(\mu), X[\ell]) \to \hom(\mathscr{O}(\mu), Y[\ell])$ in the derived category of $\mathbf{Pre}(\Sigma)$ is an isomorphism for all $\mu \in \Sigma$ and all $\ell \in \mathbb{Z}$ (here $X[\ell]$ denotes the ℓth shift of X). Equivalently, f induces isomorphisms of hyper-derived inverse limits of $X(-\mu)[\ell]$ and $Y(-\mu)[\ell]$, the ℓth shift of the μth negative twist of X and Y.

The class of colocal equivalence now turns out to do the trick. The category $\mathbf{Pre}(\Sigma)$ admits a "colocal" model structure with weak equivalences the colocal equivalences and homotopy category equivalent to the (unbounded) derived category $D(\mathfrak{Qco}(X_\Sigma))$. Moreover, the set of line bundles $\{\mathscr{O}(\mu) \mid \mu \in \Sigma\}$ is a set of weak generators of $D(\mathfrak{Qco}(X_\Sigma))$ in the sense that mapping out of these objects, in $D(\mathfrak{Qco}(X_\Sigma))$, detects isomorphisms.

For $X = \mathbb{P}^n$ a projective space the above result recovers the classical theorem that the derived category is generated by the line bundles $\mathscr{O}(k)$, $0 \leq k \leq n$.

The cofibrant objects of the colocal model structure are particularly interesting: they are characterised by a weak form of gluing condition. Instead of requiring isomorphisms as in (0.1) we ask for quasi-isomorphisms

$$A^\tau \otimes_{A^\sigma} Y^\sigma \simeq Y^\tau$$

for all pairs of cones $\tau \subseteq \sigma$ in Σ. We call the resulting structure a *homotopy sheaf*. A main ingredient of the proof is that the homotopy category of homotopy sheaves is nothing but the (unbounded) derived category of quasi-coherent sheaves on X_Σ. This stems from the fact that homotopy sheaves can be replaced, up to quasi-isomorphism on the covering sets, by quasi-coherent sheaves.

Homological invariants over the exterior algebra

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This is a report on a joint work with M. Kubitzke and T. Römer.
One can do homological algebra over the exterior algebra. There are a lot of similarities to homological algebra over the polynomial ring, but also a lot of differences. In the talk some homological invariants as e.g. regularity, complexity and depth over the exterior algebra are introduced and similarities and differences are noted. We give formulas relating these notions and apply the results on exterior face rings of simplicial complexes and on Orlik-Solomon algebras, the singular cohomology of complements of subspace arrangements interpreted as quotient ring over the exterior algebra.

1

Let K be an infinite field and $E = K\langle e_1, \ldots, e_n \rangle$ the *exterior algebra* over a vectorspace with basis $e_1, \ldots, e_n$. It is a skew-commutative algebra, equipped with a $\mathbb{Z}$-grading with $\deg e_i = 1$. The elements $e_F = e_{i_1} \wedge \ldots \wedge e_{i_s}$ for $F = \{i_1, \ldots, i_s\}$ are called *monomials*. They form a K-basis of E.

We work in the category of finitely generated graded E-modules M satisfying the condition $a \cdot m = (-1)^{(\deg a)(\deg m)} m \cdot a$ for homogeneous elements $a \in E$, $m \in M$, denoted $\mathscr{M}$. Throughout this note let M be a module in $\mathscr{M}$.

A useful functor is the duality functor

$$M^* = \mathrm{Hom}_E(M, E).$$

It is contravariant and exact on $\mathscr{M}$.

One class of modules in $\mathscr{M}$ is given by simplicial complexes.

Example 1.1. Let Δ be a simplicial complex on $[n]$, i.e. Δ is a set of subsets of $[n] = \{1, \ldots, n\}$ such that, whenever $F \in \Delta$ and $G \subseteq F$ it follows that $G \in \Delta$. The elements of Δ are called faces. The ideal $J_\Delta = (e_F : F \notin \Delta)$ is called the *(exterior) face ideal* of Δ. The corresponding quotient ring $K\{\Delta\} = E/J_\Delta$ is called the *(exterior) face ring* of Δ. Both are modules in $\mathscr{M}$. (In fact, every quotient ring E/J by a graded ideal $J \subseteq E$ is an element in $\mathscr{M}$.)

Another class of examples are the Orlik-Solomon algebras (OS-algebras for short), defined by a matroid.

Example 1.2. Let $\mathbb{M}$ be a matroid on $[n]$, i.e. $\mathbb{M}$ is a simplicial complex on $[n]$ with the additional property that for $A, B \in \mathbb{M}$ with $|A| < |B|$ there exists an element $i \in B \setminus A$ such that $A \cup \{i\} \in \mathbb{M}$. A subset in $\mathbb{M}$ is called independent, otherwise dependent. To $\mathbb{M}$ one associates the ideal $J(\mathbb{M})$

generated by

$$\partial e_S = \sum_{i=0}^{t} (-1)^i e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_t}$$

for all dependent sets $S = \{j_1, \dots, j_t\}$. The quotient ring $E/J(\mathbb{M})$ is the *Orlik-Solomon algebra* of $\mathbb{M}$ and a module in $\mathscr{M}$.

One interesting special case of an OS-algebra is defined by a hyperplane arrangement. Let $\mathscr{A} = \{H_1, \dots, H_n\}$ be a central essential complex hyperplane arrangement. The intersections of the hyperplanes define a matroid by

$$S \text{ independent } \Leftrightarrow \operatorname{codim} \bigcap_{i \in S} H_i = |S|.$$

For example let H_1, H_2, H_3 be three lines through the origin in $\mathbb{C}^2$. The only dependent set is $\{1,2,3\}$ because $\operatorname{codim} H_1 \cap H_2 \cap H_3 = 2 < 3$.

There are a lot of similarities between modules over the polynomial ring S and the exterior algebra. We only state the necessary ones for this note.

Every module $M \in \mathscr{M}$ has a minimal free graded resolution over E of the form

$$\dots \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_{2j}} \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_{1j}} \longrightarrow \bigoplus_{j \in \mathbb{Z}} E(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0$$

that is unique up to isomorphisms. The resolution is linear if the entries in the matrices representing the maps in the resolution are of degree one. The exponents $\beta_{ij}(M)$ appearing in the resolution are called the *graded Betti numbers* of M. The *regularity* is defined as over the polynomial ring, i.e.

$$\operatorname{reg}(M) = \max\{j - i : \beta_{i,j}(M) \neq 0\}.$$

The category $\mathscr{M}$ has enough injectives, in fact a module is injective if and only if it is free. Thus one can look at the minimal injective graded resolution of M, which has the same shape as the minimal free graded resolution, except that the arrows are reversed and the exponents are denoted by $\mu_{ij}(M)$ and called the *graded Bass numbers* of M. In particular M has a d-linear injective resolution if and only if $\mu_{i,j-i}(M) = 0$ for all $j \neq d$ (which is the same as to demand that the entries in the matrices are of degree one).

Example 1.3.

(i) The Eagon-Reiner theorem [3, Theorem 3] states that $K\{\Delta\}$ has a d-linear injective resolution if and only if Δ is Cohen-Macaulay, where $\dim \Delta = d - 1$.

(ii) Eisenbud, Popescu and Yuzvinsky [2, Theorem 1.1] show that an OS-algebra has always an r-linear injective resolution, where r is the rank of the matroid, i.e. the cardinality of a maximal independent set.

Resolutions over E are always infinite (unless the module is free). Therefore it is not useful to define a projective dimension over E. Instead one measures the polynomial growth of the total Betti numbers.

Definition 1.4. The *complexity* of M is

$$\operatorname{cx}_E M = \inf\{c : \beta_i(M) \leq \alpha i^{c-1} \text{ for all } i \geq 1, \alpha \in \mathbb{R}\}.$$

Each linear form $v \in E_1$ annihilates the submodule vM. Thus Aramova, Avramov and Herzog introduce in [1] the following adapted notion of a regular element.

Definition 1.5. A linear form $v \in E_1$ is called *M-regular* if $0 :_M v = vM$.

In the same article they define (in the obvious way) regular sequences and show that all maximal regular sequences have the same length. Therefore they define:

Definition 1.6. The *depth* of M is the length of a maximal M-regular sequence, denoted by $\operatorname{depth}_E M$.

Regular sequences and duality commute, in particular the following holds.

Theorem 1.7. [5, Theorem 4.3] The depth of M and the depth of its dual M^* coincide.

So far we have introduced the three invariants regularity, complexity and depth. These invariants are more or less closely related to each other.

Theorem 1.8. [1, Theorem 3.2]

$$\operatorname{depth}_E M + \operatorname{cx}_E M = n.$$

Theorem 1.9. [5, Theorems 5.3,4.1] Let J be a graded ideal in E and E/J have a d-linear injective resolution. Then

(i) $\operatorname{depth}_E(E/J) + \operatorname{reg}_E(E/J) = d$,
(ii) $H(E/J,t) = Q(t) \cdot (1+t)^{\operatorname{depth}_E(E/J)}$, $Q(t) \in \mathbb{Z}[t]$ with $Q(-1) \neq 0$.

where $H(-,t)$ is the Hilbert series.

Example 1.10. These three invariants of an OS-algebra are computed in [5]. For each matroid $\mathbb{M}$ there exists a decomposition $\mathbb{M} = \mathbb{M}_1 \oplus \cdots \oplus \mathbb{M}_k$ of $\mathbb{M}$ in its so-called connected components. Let k be the number of this components. Then

(i) $\operatorname{depth}_E(E/J(\mathbb{M})) = k$;
(ii) $\operatorname{cx}_E(E/J(\mathbb{M})) = n - k$;
(iii) $\operatorname{reg}_E(E/J(\mathbb{M})) = r - k$.

Given a simplicial complex Δ it is also interesting to look at the Stanley-Reisner ideal and ring of it. These are defined analogously to the face ideal and ring, i.e.

$$I_\Delta = (x_F : F \notin \Delta) \qquad \text{and} \qquad K[\Delta] = S/I_\Delta,$$

where $S = K[x_1, \ldots, x_n]$. In [4] the invariants of $K\{\Delta\}$ and $K[\Delta]$ are compared.

Theorem 1.11. [4, Theorem 2.4]

(i) $\operatorname{reg}_E(K\{\Delta\}) = \operatorname{reg}_S(K[\Delta])$
(ii) $\operatorname{cx}_E(K\{\Delta\}) \geq \operatorname{proj\,dim}_S(K[\Delta])$
(iii) $\operatorname{depth}_E(K\{\Delta\}) \leq \operatorname{depth}_S(K[\Delta])$

There are also some refinements of this relations proved: The difference in (ii) and (iii) is bounded above by the (common) regularity. If Δ is Cohen-Macaulay or J_Δ has a linear resolution (this is the case if and only if I_Δ has a linear resolution) then this difference is exactly the regularity.

These inequalities are the best one can obtain as the following example shows.

Example 1.12. [4, Example 2.11] Given $r,s,t \in \mathbb{N}$ with $0 \leq s-t \leq r$, then there exists a simplicial complex Δ with

$$\operatorname{depth}_S(K[\Delta]) = s, \quad \operatorname{depth}_E(K\{\Delta\}) = t \quad \text{and} \quad \operatorname{reg}_E(K\{\Delta\}) = r.$$

References

[1] A. Aramova, L. L. Avramov, and J. Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Am. Math. Soc. **352** (2000), no. 2, 579–594.
[2] D. Eisenbud, S. Popescu, and S. Yuzvinsky, *Hyperplane Arrangement Cohomology and Monomials in the Exterior Algebra*, Trans. Am. Math. Soc. **355** (2003), no. 11, 4365–4383.
[3] J. Eagon and V. Reiner, *Resolutions of Stanley-Reisner rings and Alexander duality*, Pure Appl. Algebra **130** (1998), 180 –234.
[4] G. Kämpf and M. Kubitzke, *Exterior depth and exterior generic annihilator numbers*, to appear in Commun. Alg., (2009).
[5] G. Kämpf and T. Römer, *Homological properties of Orlik-Solomon algebras*, Manuscr. Math. **129** (2009), 181–210.

An algorithm to determine semistability of certain vector bundles on projective spaces

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The results which we present here are mostly part of one chapter of the PhD-thesis [8]. For a coherent sheaf $\mathscr{E}$ of rank r on a smooth projective variety X the *slope* is defined as $\mu(\mathscr{E}) = \deg(\mathscr{E})/r$, where the degree of $\mathscr{E}$ is taken with respect to a fixed very ample line bundle $\mathscr{O}_X(1)$ on X. Then $\mathscr{E}$ is called *semistable* (*stable*) if for every coherent subsheaf $\mathscr{F} \subset \mathscr{E}$ the inequality $\mu(\mathscr{F}) \leq (<) \mu(\mathscr{E})$ holds. Despite the fact that semistability and stability of vector bundles are fundamental notions in algebraic geometry, there is no computational approach to this topic. The problem already becomes evident if we consider a smooth projective curve C of genus $g \geq 1$ and a rank-2 vector bundle $\mathscr{E}$ over it. Then, in order to determine the semistability of $\mathscr{E}$, we have to consider all line bundles $\mathscr{L} \subset \mathscr{E}$. But, the Picard group $\mathrm{Pic}(C)$, which parameterizes all line bundles on C, is in general huge.

Here we deal with a computational approach to determine semistability of vector bundles on a projective space $\mathbb{P}^N$ which is defined over a field K. Here the situation is easier, since there are only a few line bundles, i.e., $\mathrm{Pic}(\mathbb{P}^N) = \mathbb{Z}$. First of all, we may assume $N \geq 2$ since on the projective line $\mathbb{P}^1$ by the theorem of Grothendieck, every vector bundles splits as a direct sum of line bundles. Hence every semistable vector bundle of rank r on $\mathbb{P}^1$ is of the form $\bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. In particular, the stable bundles on $\mathbb{P}^1$ are just the line bundles.

In fact, on $\mathbb{P}^N$ there is the following well-known algorithmic criterion for semistability due to H.J. Hoppe. It is based on consideration of global sections of exterior powers of a given vector bundle.

Lemma 1 (Hoppe; see [6])**.** Let $\mathscr{E}$ denote a vector bundle on $\mathbb{P}^N$ over an algebraically closed field K. Then $\mathscr{E}$ is semistable if and only if for every $q < \mathrm{rk}(\mathscr{E})$ and every $k < -q\mu(\mathscr{E})$ there does not exist a non-trivial global section of $(\bigwedge^q \mathscr{E})(k)$. Moreover, if we have $\Gamma(\mathbb{P}^N, (\bigwedge^q \mathscr{E})(k)) = 0$ for every $q < \mathrm{rk}(\mathscr{E})$ and every $k \leq -q\mu(\mathscr{E})$ then $\mathscr{E}$ is stable.

Originally, due to the fact that in positive characteristic exterior powers of a semistable vector bundle are not necessarily semistable anymore, Hoppe has formulated this result in characteristic 0. But using results of Mehta, Ramanathan and Ramanan (see [11], [13]) it follows that Lemma 1 also holds in positive characteristic. The null correlation bundles on complex projective spaces $\mathbb{P}^N_{\mathbb{C}}$ for $N \geq 5$ odd illustrate that Lemma 1 gives not a necessary condition for stability.

Having this semistability criterion at hand, it remains to find a computational way to access the global sections of the exterior powers of a given vector bundle. This is still a difficult task in general, and therefore we restrict ourselves to vector bundles which are given as kernels of a surjective mapping between splitting bundles, i.e., bundles given by a short exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \bigoplus_{i=1}^{n} \mathscr{O}_{\mathbb{P}^N}(a_i) \xrightarrow{\varphi} \bigoplus_{j=1}^{m} \mathscr{O}_{\mathbb{P}^N}(b_j) \longrightarrow 0.$$

For kernel bundles the qth exterior power sits in the well-known exact sequence

$$0 \longrightarrow \bigwedge^{q} \mathscr{E} \longrightarrow \bigoplus_{A \subseteq I, |A|=q} \mathscr{O}_{\mathbb{P}^N}(\sum_{i \in A} a_i) \xrightarrow{\varphi_q} \bigoplus_{(B,j): B \subseteq I, |B|=q-1, j \in J} \mathscr{O}_{\mathbb{P}^N}((\sum_{i \in B} a_i) + b_j)$$

($I = \{1, \dots, n\}$, $J = \{1, \dots, m\}$), which is in general not surjective on the right. Moreover, the matrix describing the map φ_q can be constructed from the matrix describing the map φ. This gives together with Hoppe's criterion a semistability algorithm for kernel bundles which is suitable for a computer algebra system, since the global sections of the exterior powers can be computed with conventional Gröbner basis methods. This algorithm has been implemented in the computer algebra system CoCoA ([3]) and can be found in [8].

Despite the restriction to kernel bundles, we obtain a lot of examples. For a family $f_1, \dots, f_n$ of R_+-primary polynomials (i.e., $\sqrt{(f_1, \dots, f_n)} = R_+$) in a polynomial ring $R = K[X_0, \dots, X_N]$ over a field K we can check whether the *syzygy bundle* $\operatorname{Syz}(f_1, \dots, f_n)$ is semistable or not. Since a vector bundle is semistable if and only if its dual bundle is semistable, our algorithm is applicable to every vector bundle of homological dimension 1. For instance, every vector bundle on the projective plane $\mathbb{P}^2$ has this property by the theorem of Horrocks (see [12]). Also Steiner bundles on $\mathbb{P}^N$, introduced by Dolgachev and Kapranov in [5], have homological dimension 1.

Moreover, we have found the following example by using our algorithm and performing some extra calculations (we recall that our algorithm is in general not able to detect stability).

Proposition 2. The syzygy bundle

$$\operatorname{Syz}(X^2 - Y^2, X^2 - Z^2, XY, XZ, YZ)$$

is stable on $\mathbb{P}^2 = \operatorname{Proj} K[X,Y,Z]$. Moreover, the syzygy bundle for 5 generic quadrics in $K[X,Y,Z]$ is stable on the projective plane.

This answers a problem of P. Marques who has proved in his thesis ([10]) the stability of the syzygy bundle $\operatorname{Syz}(f_1, \dots, f_n)$ for generic forms $f_1, \dots, f_n$ of constant degree d on $\mathbb{P}^N$ except for the case $N = 2$, $n = 5$, $d = 2$. We remark that the stability of the generic syzygy bundles in the case $N = 2$, $n = 5$, $d = 2$ has independently been proven by I. Coandă in [2] but in a more complicated way.

Another interesting application of our semistability algorithm is the following (this is joint work in progress with Ralf Kasprowitz). The category $\mathscr{T}^{ss}$ of semistable vector bundles of degree 0 on $\mathbb{P}^N$ is a neutral Tannaka category with fibre functor $\omega_x : \mathscr{T}^{ss} \to \operatorname{Vect}(K)$, $\mathscr{E} \mapsto \mathscr{E}_x$ (where x is a fixed K-rational point of $\mathbb{P}^N$). To such a category one can associate an affine group scheme G, called the *Tannaka dual* of $\mathscr{T}^{ss}$. Is there a computational approach to determine the Tannaka dual $G_{\mathscr{E}}$ of the Tannaka category generated by a given semistable vector bundle of degree 0? Which groups can be realized? Using results from [9] and classical representation theory, we obtain the following non-trivial example.

Proposition/Example 3. The Tannaka dual $G_{\mathscr{S}}$ of the syzygy bundle

$$\mathscr{S} = \operatorname{Syz}(X^4 - Y^4, X^4 - Z^4, X^2Y^2, X^2Z^2, Y^2Z^2)(5)$$

on $\mathbb{P}^2$ is the Spin group $\operatorname{Sp}_2 \subset \operatorname{GL}_4$.

Moreover, we could also show that this result holds for the Tannaka dual of this bundle in the p-adic setting described in [4].

There are also further applications of our algorithm: Using restriction theorems (see [7]) one can compute examples of semistable vector bundles on hypersurfaces $X \subseteq \mathbb{P}^n$ of sufficiently large degree. In this way, using results of H. Brenner (see [1]), one obtains a computational way to find inclusion bounds for tight closure and solid closure in the homogeneous coordinate rings of such hypersurfaces.

REFERENCES

1. H. Brenner, *Tight closure and vector bundles*, Three Lectures on Commutative Algebra (J. Elias S. Zarzuela G. Colomé-Nin, T. Cortodellas Benitez, ed.), University Lecture Series, vol. 42, AMS, 2008, pp. 1–71.
2. I. Coandă, *On the stability of syzygy bundles*, Preprint (2009), arXiv:0909.4435.
3. CoCoATeam, CoCoA*: a system for doing Computations in Commutative Algebra*, Available at `http://cocoa.dima.unige.it`.
4. C. Deninger and A. Werner, *Vector bundles on p-adic curves and parallel transport*, Ann. Scient. Éc. Norm. Sup. **38** (2005), 535–597.
5. I. Dolgachev and M. Kapranov, *Arrangements of hyperplanes and vector bundles on* $\mathbb{P}^n$, Duke Math. J. **71** (1993), no. 3, 633–664.
6. H.J. Hoppe, *Generischer Spaltungstyp und zweite Chernklasse stabiler Vektorraumbündel vom Rang* 4 *auf* $\mathbb{P}^4$, Math. Z. **187** (1984), 345–360.
7. D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, Viehweg, 1997.
8. A. Kaid, *On semistable and strongly semistable syzygy bundles*, PhD-thesis, University of Sheffield, 2009, in preparation.
9. R. Kasprowitz, *Monodromiegruppen von Vektorbündeln auf p-adischen Kurven*, Dissertation, Universität Münster, 2009.
10. P. Macias Marques, *Stability and moduli spaces of syzygy bundles*, Tesi de doctorat, Universitat de Barcelona, 2009, arXiv:0909.4646.
11. V. B. Mehta and A. Ramanathan, *Homogeneous bundles in characteristic p*, Algebraic Geometry - open problems, Lect. Notes Math., vol. 997, 1982, pp. 315–320.
12. C. Okonek, M. Schneider, and H. Spindler, *Vector Bundles on Complex Projective Spaces*, Birkhäuser, 1980.
13. S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. **36** (1984), 269–291.

Poincaré series and simple complete ideals

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1. THE SETTING

This note is based on a joint work with Karlheinz Kiyek (Universität Paderborn, Germany), cf. [3]. For notations, proofs and further details concerning this introductory section, the reader should confer to [2, Chapter VII], or [5, Appendix 5].

Let $(R,\mathfrak{m},k)$ be a regular local ring of dimension two, $K := \mathrm{Quot}(R)$ the quotient field of R. If we write $\mathfrak{m} = (x,y)$, then the graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is nothing but the polynomial ring $k[\bar{x},\bar{y}]$, where $\bar{x} = x \bmod \mathfrak{m}^2$ and $\bar{y} = y \bmod \mathfrak{m}^2$.

Let $p \in k[\bar{x},\bar{y}]$ homogeneous prime ideal of height 1. Then we have $p = (\overline{f})$ for $\overline{f}$ a homogeneous irreducible polynomial of degree n. Let $f \in R$. We define the *order function* of f as

$$\mathrm{ord}_R(f) := n \quad \text{if } f \in \mathfrak{m}^n, f \notin \mathfrak{m}^{n+1}.$$

Choose now $f \in R$, $\mathrm{ord}_R(f) = n$ with $f \bmod \mathfrak{m}^{n+1} = \overline{f}$. Assuming w.l.o.g. that $\bar{x}$ does not divide $\overline{f}$, the ring $S := R\left[\frac{y}{x}\right]_{\left(x,\frac{f}{x^n}\right)}$ is a *quadratic transform* of R.

Definition 1.1. Let $R \subset S$ having the same quotient field, and $\mathfrak{a} \neq (0)$ in R. The ideal

$$\mathfrak{a}^S := x^{-n}\mathfrak{a}S$$

whenever $\mathfrak{a} \subset \mathfrak{m}^n$, $\mathfrak{a} \not\subseteq \mathfrak{m}^{n+1}$ is called the ideal transform of $\mathfrak{a}$ in S.

It is possible to define ideal transforms independent of the choice of x, but this would exceed the purpose of this note (for further details, see [2]).

2. SIMPLE COMPLETE IDEALS

2.1. **Definition and first properties.** Following Zariski's terminology, a complete ideal of R is an integrally closed ideal of R (in K). A complete ideal $\mathfrak{a}$ is said to be *simple* if: for $\mathfrak{a} = \mathfrak{b}\cdot\mathfrak{c}$, then $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$. A remarkable theorem of Zariski stays the factorisation property of complete ideals of R into powers of simple ideals: Let $\mathfrak{a}$ be a complete ideal. Then one has $\mathfrak{a} = \wp_1^{e_1}\cdot\ldots\cdot\wp_l^{e_l}$ for $\wp_i$ simple ideal, $1 \leq i \leq l$. We will mention now some other interesting properties of simple ideals.

Let $\wp \neq \mathfrak{m}$. Then there exists a unique quadratic transform S of R with $\wp^S \neq S$, with $\wp^S$ simple.

Let $\wp \neq \mathfrak{m}$ be a simple complete $\mathfrak{m}$-primary ideal of R. Such an ideal determines a sequence

(†) $$R_0 := R \subsetneqq R_1 \subsetneqq R_2 \subsetneqq \ldots \subsetneqq R_h =: S$$

such that:

- $R_1, \dots R_h$ are two-dimensional regular local rings with quotient field K,
- for each $j \in \{1, \dots, h\}$, the ring R_j is a quadratic transform of R_{j-1},
- the residue class field k_j of R_j is a finite extension of k whose degree will be denoted by $[R_j : R]$.

The sequence (†) is called the *quadratic sequence determined by* $\wp$. We say that $\wp$ is a simple ideal of rank h. If $[R_h : R] = 1$, the ideal $\wp$ is said to be residually rational.

Let S be a quadratic transform of R. The order function $\mathrm{ord}_S(\cdot)$ gives rise to a discrete valuation ν_S of rank 1, with discrete valuation ring V_S, which satisfies: $V_S/\mathfrak{m}(V_S)$ is a simple trascendental extension of $S/\mathfrak{m}(S)$.

There is a bijection between the set of simple ideals in R and the set of valuation rings of K of the second kind dominating R.

From now on we will write V and ν instead of V_S and ν_S.

2.2. **ν-ideals.** Let $\mathfrak{a} \neq (0)$ be an ideal of R. Now $\mathfrak{a}$ is finitely generated, and so $\nu(\mathfrak{a}) := \min\{\nu(z) \mid z \in \mathfrak{a} \setminus \{0\}\}$ is well-defined. An ideal $\mathfrak{a} \neq (0)$ of R is called a *ν-ideal* if $\mathfrak{a} = \{z \in R \mid \nu(z) \geq \nu(\mathfrak{a})\}$. If $\mathfrak{a}$ is a ν-ideal, then the ν-ideal $\mathfrak{a}^+ := \{z \in R \mid \nu(z) \geq \nu(\mathfrak{a}) + 1\}$ is called the *immediate ν-successor* of $\mathfrak{a}$. Zariski showed: every ν-ideal of R is a complete $\mathfrak{m}$-primary ideal.

2.3. **ν-ideals and simple ideals.** We consider the quadratic sequence determined by $\wp$. For each $j \in \{0, \dots, h\}$, there exists a unique simple ideal $\wp_j$ of R with $\wp_j^{R_j} = \mathfrak{m}(R_j)$, where $\wp_j^{R_j}$ is the transform of $\wp_j$ in R_j and $\mathfrak{m}(R_j)$ is the maximal ideal of R_j. We have a chain of simple ideals of R, namely

$$\mathfrak{m} = \wp_0 \supsetneqq \wp_1 \supsetneqq \dots \supsetneqq \wp_h = \wp,$$

and the ideals $\wp_0, \dots, \wp_{h-1}$ are ν-ideals of R; they are called the *predecessors* of $\wp$. Every power of $\wp$ is a ν-ideal and every $\mathfrak{m}$-primary ν-ideal of R is a product of these $h+1$ simple ν-ideals. From now on, let us assume that *the residue class field $k = R/\mathfrak{m}$ of R is infinite*. One shows: if $\mathfrak{a}$ is a ν-ideal, then, for every $n \in \mathbb{N}$, $\mathfrak{a}\wp^n$ is also a ν-ideal (cf. [4]). Now, for any complete $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R we denote by $\delta_\wp(\mathfrak{a})$ the non-negative integer such that $\wp^{\delta_\wp(\mathfrak{a})}$ divides $\mathfrak{a}$ but $\wp^{\delta_\wp(\mathfrak{a})+1}$ does not divide $\mathfrak{a}$. Noh also proved: Let $\mathfrak{a}$ be a non-zero proper ν-ideal of R. Let $\mathfrak{a}^+$ be its immediate ν-successor. Then one has $\lambda_R(\mathfrak{a}/\mathfrak{a}^+) = \delta_\wp(\mathfrak{a}) + 1$.

3. Value semigroup of a simple ideal

Let $\wp$ be a residually rational simple ideal. We define the value semigroup $\Gamma_\wp$ associated with $\wp$ as $\Gamma_\wp := \{\nu(z) \mid z \in R \setminus \{0\}\} \subset \mathbb{N}_0$. It is a numerical semigroup with conductor $c_\wp$. One can easily find a system of generators of $\Gamma_\wp$. We have $h = \mathrm{rank}(\wp)$, and we set $e := \nu(\wp)$. Let $\wp_0 = \mathfrak{m}, \wp_1, \dots, \wp_{h-1}$ be the predecessors of the ideal $\wp = \wp_h$. We set $s_i := \nu(\wp_i)$ for $i \in \{0, \dots, h-1\}$, $s_h := e$; then $\Gamma_\wp$ is generated by $s_0, s_1, \dots, s_h$. But this is not a *minimal* system of generators of $\Gamma_\wp$. By using the methods of [1], one can construct a minimal system of generators $\{\rho_0, \dots, \rho_g\}$ of $\Gamma_\wp$ such that

(i) setting $\theta_1 := \rho_0$, $\theta_{i+1} := \gcd(\rho_0,\dots,\rho_i)$ for $i \in \{1,\dots,g\}$ and $n_i := \frac{\theta_i}{\theta_{i+1}}$ for $i \in \{1,\dots,g\}$, we have

$$\theta_1 > \theta_2 > \cdots > \theta_{g+1} = 1,\ \rho_{i+1} > n_i\rho_i \quad \text{for } i \in \{1,\dots,g-1\}, \tag{$*$}$$

(every numerical semigroup satisfying $(*)$ is said to have a *strong growth*).

(ii) every $\gamma \in \Gamma_{\wp}$ has a unique representation $\gamma = u_0\rho_0 + u_1\rho_1 + \cdots + u_g\rho_g$ with non-negative integers $u_0, u_1, \dots, u_g$ and $u_i < n_i$ for $i \in \{1,\dots,g\}$.

4. Poincaré Series of a numerical semigroup

A first way to define a generating series from a numerical semigroup is the following:

Definition 4.1. The Poincaré series associated with a numerical semigroup Γ is the formal power series

$$P_\Gamma(t) := \sum_{n\in\Gamma} t^n \in \mathbb{Z}[[t]].$$

(In particular we may take $\Gamma = \Gamma_{\wp}$). This series is a rational function if Γ has a strong growth, and we know an expression in terms of a minimal system of generators:

Proposition 4.2. Let Γ be a numerical semigroup having a strong growth. Consider a minimal system of generators $\{\rho_0, \rho_1, \dots, \rho_g\}$ of the semigroup Γ. The Poincaré series $P_\Gamma(t)$ is a rational function, and we have

$$P_\Gamma(t) = \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}}.$$

5. Poincaré Series of a simple ideal

We can also define a different Poincaré series for a simple ideal $\wp$ of R (related to the previous one, as we will see in Proposition 5.7) using the specific properties of the simple ideals. Let us first introduce some notations. For every $n \in \mathbb{N}_0$ we set $I_{\wp}(n) := \{z \in R \mid \nu(z) \geq n\}$, which is a ν-ideal of R; notice that, if $n \in \Gamma_{\wp}$, then $\nu\left(I_{\wp}(n)\right) = n$. Since $\nu(\mathfrak{m}I_{\wp}(n)) \geq n+1$, we have $\mathfrak{m}I_{\wp}(n) \subset I_{\wp}(n+1)$, hence the length

$$d_{\wp}(n) := \lambda_R\left(I_{\wp}(n)/I_{\wp}(n+1)\right) = \dim_k\left(I_{\wp}(n)/I_{\wp}(n+1)\right)$$

is finite. Furthermore, we set $\delta_{\wp}(n) := \delta_{\wp}(I_{\wp}(n))$.

Definition 5.1. Let $\wp$ be a simple ideal of R. The formal power series

$$P_{\wp}(t) := \sum_{n\in\mathbb{N}_0} d_{\wp}(n)t^n \in \mathbb{Z}[[t]]$$

is called the *Poincaré series* of $\wp$.

Lemma 5.2. For every $n \in \mathbb{N}_0$ we have $d_{\wp}(n) = 0$ if $n \notin \Gamma_{\wp}$, and $d_{\wp}(n) = \delta_{\wp}(n) + 1$ if $n \in \Gamma_{\wp}$.

Let $n \in \Gamma_{\wp}$. We can express n as a linear combination of the elements $s_0, \dots, s_h$ with coefficients in $\mathbb{N}_0$. We choose a combination where the coefficient of $s_h = \nu(\wp) = e$ is maximal, i.e., we write $n = z_0(n)s_0 + \cdots + z_{h-1}(n)s_{h-1} + z_h(n)s_h$ with $z_i(n) \in \mathbb{N}_0$ and $z_h(n)$ maximal.

Lemma 5.3. Let $n \in \mathbb{N}_0$. Then we have

$$z_h(n) = \delta_{\wp}(n).$$

Let $\kappa \in \mathbb{N}_0$. We set $A_\kappa := \{\gamma \mid \gamma \in \Gamma_{\wp},\ \gamma - \kappa e \in \Gamma_{\wp}\} \subseteq \mathbb{N}_0$, and $\chi_{A_\kappa} : \mathbb{N}_0 \to \{0,1\}$ the characteristic function of A_κ.

Proposition 5.4. Let $n \in \Gamma_{\wp}$. Then we have

$$\sum_{\kappa \in \mathbb{N}_0} \chi_{A_\kappa}(n) = \delta_{\wp}(n) + 1.$$

Definition 5.5. The ideal $\wp$ is said to be an s-ideal if S is proximate to R_i for some $i \in \{0, \ldots, h-2\}$.

Theorem 5.6. (1) If $\wp$ is not an s-ideal, then

$$P_{\wp}(t) = \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}} \cdot \frac{1}{1-t^e}.$$

(2) If $\wp$ is an s-ideal, then

$$P_{\wp}(t) = \frac{1}{1-t^{\rho_0}} \cdot \prod_{i=1}^{g-1} \frac{1-t^{n_i\rho_i}}{1-t^{\rho_i}} \cdot \frac{1}{1-t^{\rho_g}}.$$

In particular, the Poincaré series $P_{\wp}(t)$ is a rational function.

Proposition 5.7. We have

$$P_{\wp}(t) = (1-t^e)^{-1} P_{\Gamma_{\wp}}(t).$$
$$P_{\wp}(t^{-1}) = t^{e-c_{\wp}+1} \cdot P_{\wp}(t).$$
$$P_{\Gamma_{\wp}}(t^{-1}) = -t^{-c_{\wp}+1} \cdot P_{\Gamma_{\wp}}(t).$$

Proposition 5.8. We can express the Poincaré series of $\wp$ as

$$P_{\wp}(t) = \frac{\Lambda_{\wp}(t)}{(1-t)(1-t^e)},$$

with $\Lambda_{\wp}(t) := t^{c_{\wp}} + (1-t)\sum_{\substack{n \in \Gamma_{\wp} \\ n < c_{\wp}}} t^n$.

Corollary 5.9. The following data are equivalent: (1) The Poincaré series $P_{\wp}(t)$ and the multiplicity $e = \nu(\wp)$; (2) the Poincaré series $P_{\Gamma_{\wp}}(t)$; (3) the value semigroup $\Gamma_{\wp}$.

REFERENCES

[1] S. Greco, K. Kiyek: *General elements of Complete Ideals and Valuations Centered at a Two-dimensional Regular Local Ring*. In: Christensen et al. (Eds.): Algebra, Arithmetic and Applications. Springer, Berlin-Heidelberg-New York, 2003.

[2] K. Kiyek, J. L. Vicente: *Resolution of Curve and Surface Singularities in Characteristic Zero*. Kluwer, Dordrecht, 2004.

[3] K. Kiyek, J.J. Moyano-Fernández: *The Poincaré series of a simple complete ideal of a two-dimensional regular local ring*. Journal of Pure and Applied Algebra **213** (2009) 1777–1787.

[4] S. Noh: *The value-semigroups of prime divisors of the second kind in* 2*-dimensional regular local rings*. Trans. Amer. Math. Soc. **336** (1993) 607–619.

[5] O. Zariski, P. Samuel: *Commutative Algebra. Volume II*. Springer-Verlag, New York, 1960.

The shape of a pure O-sequence

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First of all, I would like to thank H. Brenner, W. Bruns, T. Römer, O. Röndigs, and R. Vogt for organizing the inspirational conference "Combinatorial Structures in Algebra and Topology," and for inviting me to give a talk. Its material is based on joint work with M. Boij, J. Migliore, R. M. Miró-Roig, and F. Zanello [1].

The problem of characterizing pure O-sequences dates back to Stanley's 1977 paper [6]. This fascinating and difficult problem lies at the crossroads of the combinatorics of simplicial complexes, the theory of matroids, and commutative algebra.

Recall that a non-empty set M of monomials in the variables $x_1,\dots,x_r$ is called an *order ideal (of monomials)* if whenever m is in M and m' is a monomial dividing m, then m' is in M. Ordering M partially by divisibility, its maximal elements are called the generators of M. If all generators have the same degree, say e, then M is called a *pure order ideal*. Its *h-vector* is the vector $\underline{h} = (h_0,h_1,...,h_e)$, where h_i is the number of monomials of degree i in M. A *pure O-sequence* is the h-vector of a pure order ideal.

Example 1. The order ideal in three variables x,y,z generated by x^3,y^2z is

$$\begin{aligned} M = \{&x^3,y^2z,\\ &x^2,y^2,yz,\\ &x,y,z,\\ &1\}. \end{aligned}$$

Its h-vector is

$$\underline{h} = (1,3,3,2).$$

Pure O-sequences also have an algebraic description. To this end consider quotients of the polynomial ring $S = K[x_1,\dots,x_r]$ over an arbitrary field K with its standard grading, where every variable has degree one. If $I \subset S$ is an ideal that can be generated by homogeneous polynomials, then $A := S/I$ is a graded K-algebra $A = \oplus_{j\geq 0}[A]_j$. Its Hilbert function is

$$h_A : \mathbb{N}_0 \to \mathbb{N}_0,\ j \mapsto \dim_K [A]_j.$$

A ring A is called artinian if A is a finite-dimensional K-vector space. Recording in this case only the finitely many positive values gives its h-vector $\underline{h} = (h_0,h_1,...,h_e)$. It is called an O-sequence. The possible O-sequences are classified by a result of Macaulay that gives a combinatorial description of them (see, e.g., [2]).

A ring $A = S/I$ is said to be *monomial* if I can be generated by monomials. The *socle* of A is

$$\operatorname{Soc} A := \{f \in A : f \cdot (x_1,\dots,x_r) = 0\}$$

An artinian K-algebra A is said to be *level* if its socle is concentrated in one degree, which is then called the *socle degree* of A. Algebraically, pure O-sequences are the h-vectors $\underline{h} = (h_0,h_1,...,h_e)$

of artinian monomial level algebras. The order ideal corresponding to I giving the same pure O-sequence is the complement of I in the set of monomials in S.

Example 2. The order ideal M in Example 1 corresponds to the algebra

$$A = K[x,y,z]/(x^4,y^3,xy,xz,z^2).$$

An interesting class of pure O-sequences is derived from simplicial complexes. A simplicial complex Δ on the set $[r] := \{1,\ldots,r\}$ is a collection of subsets of $[r]$ that is closed under inclusion. Its facets are the maximal subsets. If the facets of Δ all have the same number of elements, say $e+1$, then Δ is said to be *pure*. Identifying in this case, the facets with monomials in S, then the f-vector of Δ is the h-vector of the order ideal generated by the facets. Thus, the f-vector of Δ is a pure O-sequence $\underline{h} = (h_0,h_1,...,h_e)$.

Another conjectural source of pure O-sequences is related to matroids. A *matroid complex* is a simplicial complex Δ such that Δ and all of its restrictions are pure. Equivalently, the facets of a matroid complex correspond to the maximal independent sets of a matroid. Using its f-vector $(f_{-1}, f_0, \ldots, f_e)$, its h-vector $\underline{h} = (h_0,h_1,\ldots)$ is defined by comparing coefficients

$$\sum_j h_j t^i = \sum_{j=0}^{e} f_{j-1} t^j (1-t)^{e-j}.$$

An affirmative answer to the following 1977 conjecture of Stanley would greatly contribute to the problem of describing the f-vectors of matroid complexes.

Conjecture 3 (Stanley [6]). The h-vector of every matroid complex is a pure O-sequence.

This conjecture is wide open.

Hopefully, the above examples illustrate that it is very interesting to if not characterize then at least to describe restrictions on the set of possible pure O-sequences. A main result in this regard is due to Hibi.

Theorem 4 (Hibi [5], Theorem 1.1). Let $\underline{h} = (h_0,h_1,...,h_e)$ be a pure O-sequence of socle degree e. Then

$$h_i \le h_j$$

whenever $0 \le i \le j \le e-i$.

This implies in particular that the “first half” of $\underline{h}$ is non-decreasing: $1 = h_0 \le h_1 \le h_2 \le \cdots \le h_{\lceil \frac{e}{2} \rceil}$. Hausel strengthened these estimates in 2005.

Theorem 5 (Hausel [4], Theorem 6.2). If $\underline{h} = (h_0,h_1,...,h_e)$ is a pure O-sequence of socle degree e, then

$$(1, h_1 - 1, h_2 - h_1, \ldots, h_{\lceil \frac{e}{2} \rceil} - h_{\lceil \frac{e}{2} \rceil - 1})$$

is an O-sequence, i.e. the “first half” of $\underline{h}$ is a *differentiable O-sequence*.

The converse of this is true as well, so that we have a complete characterization of the “first half” of a pure O-sequence.

Theorem 6 ([1]). An O-sequence $\underline{h}$ is the “first half” of a pure O-sequence if and only if it is differentiable.

Since the truncation of a pure O-sequence is again a pure O-sequence, this implies:

Corollary 7. Any finite differentiable O-sequence $\underline{h}$ is pure.

Differentiable O-sequences are necessarily non-decreasing. Using an upper bound for $h_i - h_{i-1}$ in terms of $h_2 - h_1$ whenever $2 \le i \le e$ and $(h_0, h_1, ..., h_e)$ is a pure O-sequence, one gets the following partial converse of Corollary 7.

Theorem 8 ([1]). Any non-decreasing pure O-sequence $\underline{h}$ of socle degree $e \le 3$ is differentiable.

Examples in [1] show that the analogous result is not true whenever $e \ge 4$. For example, $(1, 8, 16, 24, 36)$ is a pure O-sequence that is not differentiable.

Hibi's result implies that a pure O-sequence $\underline{h} = (h_0, h_1, ..., h_e)$ is *unimodal*, i.e., it does not have strict increase after any weak decrease, whenever $e \le 3$. Stanley [6] gave the first example of a non-unimodal pure O-sequence: $(1, 505, 2065, 3395, 3325, 3493)$. This sequence has one "valley." Similar examples are produced in [1] whenever $e \ge 4$. However, much more is true.

Theorem 9 ([1]). Let m be any positive integer, and fix an integer $h_1 \ge 3$. Then there exists a pure O-sequence $(h_0, h_1, h_2, ..., h_e)$, which is non-unimodal, having exactly m "valleys."

Hausel's method for showing Theorem 5 is related to the following property whose name is motivated by the Hard Lefschetz theorem.

Definition 10. A graded artinian algebra A has the *Weak Lefschetz property (WLP)* if there is a linear form $\ell \in [A]_1$ such that, for all integers j, the multiplication map

$$\times \ell : [A]_j \to [A]_{j+1}$$

has maximal rank, i.e., it is injective or surjective. It is said that A has the *Strong Lefschetz property (SLP)* if there is an $\ell \in [A]_1$ such that

$$\times \ell^s : [A]_j \to [A]_{j+s}$$

has maximal rank for all j and all s.

If the field K has characteristic zero, then any artinian algebra in at most two variables has the SLP (see [3]). Restricting to monomial algebras, one has:

Theorem 11. Let A be an artinian monomial level K-algebra with h-vector $(h_0, h_1, \ldots, h_e)$, where K has characteristic zero. Then:

(a) ([7], [8]) If $h_e = 1$, then A has the SLP.
(b) ([1]) If $h_e = 2$ and $h_1 \le 3$, then A has the WLP.

Furthermore, Examples in [1] show that the SLP may fail whenever $h_e \ge 2$, $h_1 \ge 3$, and e is sufficiently large. Thus, the theorem cannot be extended any further. Notice that Part (a) is the key to Hausel's Theorem 5 that led to a characterization of the first half of a pure O-sequence. However, the above examples indicate that a full characterization of pure O-sequences seems extremely difficult. This motivates the proposal of the following conjecture that seems more manageable, but stills gives structural information. It is analogous to a recent conjecture for arbitrary artinian level algebras by Zanello [9].

Conjecture 12 (Interval Conjecture for Pure O-sequences [1]). Assume that, for some integer $\alpha > 0$, both $(1, h_1, ..., h_i, ..., h_e)$ and $(1, h_1, ..., h_i + \alpha, ..., h_e)$ are pure O-sequences, then, for each integer $\beta = 0, 1, ..., \alpha$, also $(1, h_1, ..., h_i + \beta, ..., h_e)$ is a pure O-sequence.

This conjecture is known only in a few cases, most notably if the socle degree is small.

Theorem 13 ([1]). The Interval Conjecture is true for all pure O-sequences (h_0, h_1, h_2, h_3).

While this result still does not give a complete classification of all pure O-sequences of length four, in combination with some other ideas it is strong enough to count their number asymptotically

Theorem 14 ([1]). Denote by $p(t)$ the number of pure O-sequences $(1, h_1, h_2, t)$. Then

$$\lim_{t \to \infty} \frac{p(t)}{t^2} = \frac{9}{2}.$$

References

[1] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel, and F. Zanello, *On the shape of a pure O-sequence* (in preparation).

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[3] T. Harima J. Migliore, U. Nagel and J. Watanabe: *The Weak and Strong Lefschetz properties for artinian K-algebras*, J. Algebra **262** (2003), 99–126.

[4] T. Hausel, *Quaternionic geometry of matroids*, Cent. Eur. J. Math. **3** (2005), 26–38.

[5] T. Hibi: *What can be said about pure O-sequences?*, J. Combin. Theory Ser. A **50** (1989), 319–322.

[6] R. Stanley, *Cohen-Macaulay Complexes*, Higher Combinatorics (M. Aigner Ed.), Reidel, Dordrecht and Boston, 1977, 51–62.

[7] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184.

[8] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. 11, Kinokuniya Co. North Holland, Amsterdam (1987), 303–312.

[9] F. Zanello, *Interval Conjectures for level Hilbert functions*, J. Algebra **321** (2009), 2705–2715.

Algebraic K-theory and Diagrams of Spaces

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1. INTRODUCTION

In his foundational paper [11], Waldhausen introduced an abstract version of K-theory that generalised Quillen's algebraic K-theory to categories of non-abelian origin. The input of Waldhausen's K-theory is a category with cofibrations and weak equivalences, now known as a Waldhausen category. The definition of Waldhausen's K-theory allowed to extend algebraic K-theory to categories of spaces. In particular, Waldhausen defined the algebraic K-theory $A(X)$ of a space X to be the K-theory of a Waldhausen category of retractive spaces over X.

The purpose of this short announcement is to describe how the algebraic K-theory of a space X can be identified in a natural way with the K-theory of a Waldhausen category of diagrams of based simplicial sets indexed by a suitable poset. A more detailed account of the results here will appear in [8].

We briefly sketch the basic idea. Let $f : Y \to Y'$ be a map of retractive spaces over X and suppose that there is a poset $\mathscr{P}_X$ of simplices of X such that $B\mathscr{P}_X \simeq X$. If we think of every object σ of $\mathscr{P}_X$ as a simplex in X, then we can consider the restriction of the map f over that simplex σ. This way we obtain diagrams of simplicial sets $\mathscr{P}_X(Y), \mathscr{P}_X(Y') : \mathscr{P}_X \to \mathbf{SSet}$ associated with the retractive spaces Y and Y' over X, together with a natural transformation $\mathscr{P}_X(f)$ between them that is induced by the restrictions of the map f. Furthermore, by quotienting out the image of σ from $\mathscr{P}_X(Y)(\sigma)$, we can assume that the diagrams are based.

There is a Waldhausen structure on the category of such diagrams of based simplicial sets where the weak equivalences are defined pointwise. It can be shown that the K-theory of these Waldhausen categories of diagrams produces a model for $A^{\%}(X)$, where $A^{\%}$ denotes the excisive approximation to A-theory [12], [11]. This agrees intuitively with the expectation that the excisive approximation to A-theory should be the space of weak homotopy equivalences of based simplicial sets parametrised by the space X.

The original map of retractive spaces $f : Y \to Y'$ can be canonically recovered up to homotopy as the map between the (unbased) homotopy colimits of the diagrams $\mathscr{P}_X(Y)$ and $\mathscr{P}_X(Y')$ that is induced by $\mathscr{P}_X(f)$. For homotopical purposes, it turns out that a Waldhausen category of retractive spaces can be replaced by a Waldhausen category of diagrams of based simplicial sets indexed by a suitable poset and where the weak equivalences are defined to be the *global* weak equivalences, i.e. the natural transformations that induce a weak homotopy equivalence between the unbased homotopy colimits. From this, it follows that the K-theory of this Waldhausen category of diagrams produces a model for $A(X)$.

In order to complete the picture, it remains to see that one can do homotopy theory in the category of posets and that this homotopy theory models the homotopy theory of spaces.

2. Homotopy Theory of Posets

The most successful and widely applied method of introducing a homotopy theory in a category is by endowing it with a Quillen model structure [7]. More recent references for the theory of model categories are [2] and [3].

Let **Cat** denote the category of small categories and functors. A morphism $F : C \to D$ in **Cat** is called a weak equivalence if it induces a weak homotopy equivalence between the nerves $NF : NC \to ND$.

Thomason [10] proved that the class of weak equivalences is part of a model structure on **Cat**. Moreover, there is a Quillen equivalence:

$$catSd^2 : \mathbf{SSet} \rightleftarrows \mathbf{Cat} : Ex^2N$$

where $cat : \mathbf{SSet} \to \mathbf{Cat}$ denotes the left adjoint of the nerve functor. Let **Pos** denote the full subcategory of **Cat** with objects the posets. The left Quillen equivalence $catSd^2 : \mathbf{SSet} \to \mathbf{Cat}$ factors through the inclusion $i : \mathbf{Pos} \to \mathbf{Cat}$. Let $pos : \mathbf{Cat} \to \mathbf{Pos}$ denote the left adjoint to the inclusion functor.

Theorem 2.1. Thomason's model structure on **Cat** restricts to a proper cofirantly generated model structure on **Pos**. Moreover, the adjunction

$$pos : \mathbf{Cat} \rightleftarrows \mathbf{Pos} : i$$

is a Quillen equivalence.

Every cofibration in **Pos** is a Dwyer morphism.[1] Recall that a morphism $F : C \to D$ in **Cat** is a Dwyer morphism if it is the inclusion of a sieve and there is a cosieve C' in D that contains C such that the inclusion $C \to C'$ has a right adjoint. It is possible to show that there is also a (proper, cofibrantly generated) model structure on **Cat** (resp. **Pos**) with the same weak equivalences and where every pseudo-Dwyer morphism between finite categories (resp. posets) is a cofibration.

The category of posets is isomorphic with the category of Alexandroff T_0-spaces. A topological space is called Alexandorff if every intersection of open sets is open. For example, a topological space with finitely many points is Alexandroff. The homotopy theory of finite topological spaces has been studied in [9], [4]. It turns out that every (finite) simplicial complex is weakly homotopy equivalent to a (finite) Alexandroff T_0-space [6], [5]. Under the correspondence between posets and Alexandroff T_0-spaces, a Dwyer morphism between posets corresponds to an inclusion of a neighborhood deformation retract. Moreover, a morphism between posets is a weak equivalence if and only if the corresponding map between Alexandroff spaces is a weak homotopy equivalence.

3. Algebraic K-theory of spaces via diagrams

Let C be a small category. There is a model structure on the category of C-diagrams of based simplicial sets $\mathbf{SSet}_*^C$ where the weak equivalences and the fibrations are defined pointwise [2].

[1] Note that a cofibration in **Cat** is only a pseudo-Dwyer morphism in general [1]. But a pseudo-Dwyer morphism between posets is always a Dwyer morphism.

Furthermore, for every morphism $u : C \to D$ in **Cat**, there is a natural Quillen adjunction

$$u_* : \mathbf{SSet}_*^C \rightleftarrows \mathbf{SSet}_*^D : u^*$$

where u^* denotes the pullback functor.

Let $\mathbf{Pos}^{fin}$ denote the full subcategory of **Pos** with objects the finite posets. For every finite poset J, let $\mathscr{D}ia(J, \mathbf{SSet}_*)$ denote the full subcategory of $\mathbf{SSet}_*^J$ with objects the finitely presentable cofibrant objects. The model structure on $\mathbf{SSet}_*^J$ induces a Waldhausen structure on $\mathscr{D}ia(J, \mathbf{SSet}_*)$ and, for every morphism $u : I \to J$ between finite posets, the left Quillen functor $u_* : \mathbf{SSet}_*^I \to \mathbf{SSet}_*^J$ restricts to an exact functor $u_* : \mathscr{D}ia(I, \mathbf{SSet}_*) \to \mathscr{D}ia(J, \mathbf{SSet}_*)$ between Waldhausen categories. Hence there is a functor

$$K(-, \mathbf{SSet}_*) : \mathbf{Pos}^{fin} \to \mathbf{Top}, J \mapsto K(\mathscr{D}ia(J, \mathbf{SSet}_*)).$$

A functor $F : \mathbf{Pos}^{fin} \to \mathbf{Top}$ is called excisive if it sends pushout squares in $\mathbf{Pos}^{fin}$

$$\begin{tikzcd} I \arrow[r] \arrow[d, "i"] & P \arrow[d] \\ J \arrow[r] & Q \end{tikzcd}$$

where i is a Dwyer morphism, to homotopy cartesian squares

$$\begin{tikzcd} F(I) \arrow[r] \arrow[d] & F(P) \arrow[d] \\ F(J) \arrow[r] & F(Q) \end{tikzcd}$$

in **Top**. If F also sends weak equivalences to weak homotopy equivalences, then it is called homological. The terminology here follows the analogous terminology of [11] for functors defined on the category of simplicial sets.

It turns out that the functor $K(-, \mathbf{SSet}_*)$ is excisive, but it is not homotopy invariant. In fact, it can be shown that there is a (non-canonical) homotopy equivalence $K(J, \mathbf{SSet}_*) \simeq K(\star, \mathbf{SSet}_*)^{|J|}$ where $|J|$ denotes the cardinality of (the set of objects of) J. However, given an excisive functor F, there is natural way of defining a functor $\overline{F}$ that is the (unique up to homotopy) homological functor associated with $F(\star)$. The construction is technical and it will be omitted from here. It is inspired by the analogous construction for excisive functors on the category of simplicial sets [11].

A natural tranformation $\eta : F \to G$ in $\mathscr{D}ia(J, \mathbf{SSet}_*)$ is called a global weak equivalence if it induces a weak homotopy equivalence

$$hocolim(\eta) : hocolim F \to hocolim G$$

between the unbased homotopy colimits. The class of global weak equivalences defines a new Waldhausen structure on $\mathscr{D}ia(J, \mathbf{SSet}_*)$ where the cofibrations are the same as before. This new

Waldhausen category will be denoted by $\mathscr{D}ia_{global}(J, \mathbf{SSet}_*)$. Note that, for every morphism $u : I \to J$ between finite posets, the functor u_* is exact, i.e. it preserves the global weak equivalences. Hence there is a functor

$$K_{global}(-, \mathbf{SSet}_*) : \mathbf{Pos}^{fin} \to \mathbf{Top}, J \mapsto K(\mathscr{D}ia_{global}(-, \mathbf{SSet}_*)).$$

Theorem 3.1. There is a natural homotopy equivalence $A(NJ) \simeq K_{global}(J, \mathbf{SSet}_*)$.

The functors $K(-, \mathbf{SSet}_*)$ and $K_{global}(-, \mathbf{SSet}_*)$ can be extended to all posets such that they preserve directed colimits. For example, we define $K(J, \mathbf{SSet}_*)$ to be

$$colim_{J_\lambda \subset J} K(J_\lambda, \mathbf{SSet}_*)$$

where the colimit is taken over all the finite subposets $J_\lambda \subset J$ ordered by inclusion.

Note that a pointwise weak equivalence is also a global weak equivalence, i.e. the identity functor $\mathscr{D}ia(J, \mathbf{SSet}_*) \to \mathscr{D}ia_{global}(J, \mathbf{SSet}_*)$ is exact. This induces a natural map $\overline{K}(J, \mathbf{SSet}_*) \to K_{global}(J, \mathbf{SSet}_*)$ which can be identified with the assembly map in A-theory [12],[11].

More generally, we can consider Waldhausen categories of diagrams with values in a model category (or Waldhausen category) other than based simplicial sets. The notation $K(J, \mathbf{SSet}_*)$ and $K_{global}(J, \mathbf{SSet}_*)$ is meant to suggest that $\mathbf{SSet}_*$ may be regarded as the category of "coefficients" for the K-theory of J. More specifically, for a nice pointed model category $\mathscr{M}$, we can proceed similarly to define an excisive functor $K(-, \mathscr{M})$, a homological functor $\overline{K}(-, \mathscr{M})$ and a homotopy invariant functor $K_{global}(-, \mathscr{M})$ that can be intuitively understood as the Waldhausen K-theory of a category of retractive "spaces" over NJ whose fibers are in $\mathscr{M}$.

REFERENCES

[1] CISINSKI, DENIS-CHARLES, *La classe des morphismes de Dwyer n'est pas stable par retractes*, Cahiers Topologie Géom. Différentielle Catég. 40 (1999), no. 3, 227–231.

[2] HIRSCHHORN, PHILIP S., *Model categories and their localizations*, Mathematical Surveys and Monographs, Vol. 99. American Mathematical Society, Providence, RI, 2003.

[3] HOVEY, MARK *Model categories*, Mathematical Surveys and Monographs, Vol. 63. American Mathematical Society, Providence, RI, 1999.

[4] MAY, PETER J., *Finite topological spaces*, Notes for REU, available at http://www.math.uchicago.edu/ ∼ may/.

[5] MAY, PETER J., *Finite spaces and simplicial complexes*, Notes for REU, available at http:// www. math.uchicago.edu/ ∼ may/.

[6] MCCORD, MICHAEL C., *Singular homology groups and homotopy groups of finite topological spaces*, Duke Math. J. 33 (1966), 465–474.

[7] QUILLEN, DANIEL G., *Homotopical algebra*, Lecture Notes in Math., No. 43. Springer-Verlag, Berlin-New York 1967.

[8] RAPTIS, GEORGE, *Homotopy theory of posets and algebraic K-theory of spaces*, in preparation.

[9] STONG, ROBERT E., *Finite topological spaces*, Trans. Amer. Math. Soc. 123 (1966), 325–340.

[10] THOMASON, ROBERT W., *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle 21 (1980), no. 3, 305–324.

[11] WALDHAUSEN, FRIEDHELM, *Algebraic K-theory of spaces*, Lecture Notes in Math., No. 1126, 318-419, Springer, Berlin, 1985.

[12] WEISS, MICHAEL; WILLIAMS, BRUCE, *Assembly* [Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)], London Math. Soc. Lecture Note Ser., No. 227, 332-352, Cambridge Univ. Press, Cambridge, 1995.

An update on the Hirsch conjecture

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This is a report on a joint work with Edward D. Kim, University of California, Davis, USA. The Hirsch conjecture was posed in 1957 in a letter from Warren M. Hirsch to George Dantzig. It states that the graph of a d-dimensional polytope with n facets cannot have diameter greater than $n-d$. That is to say, we can go from any vertex to any other vertex using at most $n-d$ edges.

Despite being one of the most fundamental, basic and old problems in polytope theory, what we know is quite scarce. Most notably, no polynomial upper bound is known for the diameters of polytopes. In contrast, very few polytopes are known where the bound $n-d$ is attained. This talk collected several results and remarks both on the positive and on the negative side of the conjecture, with special emphasis on constructions attempted to *disprove* the conjecture.

This write-up has two parts. The first section is an extended abstract of our survey on the Hirsch conjecture [12]. A previous, also excellent, survey on the topic is [14]. Section 2 concentrates on classic counter-examples to three slight generalizations of the main conjecture. We have included these constructions in detail since we believe they deserve more attention, and they could lead to counterexamples to the Hirsch conjecture itself.

1. QUICK REVIEW ON THE HIRSCH CONJECTURE

Convex polytopes generalize convex polygons (of dimension two). More precisely, a *convex polyhedron* is any intersection of finitely many affine semi-spaces in $\mathbb{R}^d$. A polytope is a bounded polyhedron. From the applications point of view, a polyhedron is the *feasibility region* of a *linear program* [4]. The long-standing Hirsch conjecture is the following very basic statement about the structure of arbitrary polytopes. Besides its implications in linear programming, which motivated the conjecture in the first place, it is one of the most fundamental open questions in polytope theory.

Conjecture 1.1 (Hirsch conjecture)**.** Let $n > d \geq 2$. Let P be a d-dimensional polytope with n facets. Then $\operatorname{diam}(G(P)) \leq n-d$.

The number $\operatorname{diam}(G(P)) \in \mathbb{N}$ is the *diameter of the graph of P*. Put differently, the conjecture states that we can go from any vertex of P to any other vertex traversing at most $n-d$ edges. *Facets* are the faces of dimension $d-1$ of P, so that the number n of them is the minimum number of semi-spaces needed to produce P as their intersection.

Linear Programming. In linear programming, one is given a system of linear equalities and inequalities, and the goal is to maximize (or minimize) a certain linear functional. In its standard form, a linear program is given as an $m \times n$ real matrix A, and two real vectors $\mathbf{b} \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. The problem is

$$\text{Maximize } c \cdot \mathbf{x}, \text{ subject to } A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0.$$

Suppose the matrix A has full row rank $m \leq n$. Then, the equality $A\mathbf{x} = \mathbf{b}$ defines a d-dimensional affine subspace ($d = n - m$), whose intersection with the linear inequalities $\mathbf{x} \geq 0$ gives the *feasibility polyhedron* P:

$$P := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq 0\}.$$

Polynomial time algorithms for linear programming are known since thirty years ago [11, 10]. Yet, still to this day the most commonly used method for linear programming is the *simplex method* devised by G. Dantzig in 1947. In geometric terms, this method first finds an arbitrary vertex in the feasibility polyhedron. Then, it uses local rules to move from vertex to adjacent vertex in such a way that the value $c \cdot \mathbf{x}$ of the linear functional increases at every step. When there is no such *pivot step* that can increase the functional, convexity implies that we have arrived to the maximum possible value of it.

The complexity of the simplex method depends on the local *pivot rule* chosen to move from vertex to vertex. Several rules have been proposed, but for most of them it has been proved that they can lead to paths of exponential length [13]. There are subexponential, but yet not polynomial, randomized pivot algorithms. However, the simplex algorithm is highly efficient *in practice* on most linear optimization problems.

There is another reason why investigating the complexity of the simplex method is important. The algorithms known are polynomial in the *bit length* of the input; but it is of practical importance to know whether a polynomial algorithm for linear programming in the *real number machine* model exists. That is, is there an algorithm that uses a polynomial number of arithmetic operations on the coefficients of the linear program, rather than on their bits; or, better yet, a *strongly polynomial algorithm*, i.e. one that is polynomial both in the arithmetic sense and the bit sense? These two related problems were included by Smale in his list of "mathematical problems for the next century" [18]. A polynomial pivot rule for the simplex method would solve them in the affirmative.

In this context, the following *polynomial version* of the conjecture is relevant, if the linear one turns out to be false.

Conjecture 1.2 (Polynomial Hirsch conjecture)**.** Is there a polynomial function $f(n,d)$ such that for any polytope (or polyhedron) P of dimension d with n facets, $\mathrm{diam}(G(P)) \leq f(n,d)$?.

Simple polytopes suffice. Of special importance are the simple and simplicial polytopes. A d-polytope is called *simple* if every vertex is the intersection of exactly d facets. Equivalently, if every vertex in the graph $G(P)$ has degree exactly d. Any polytope or polyhedron P, given by its facet-description, can be perturbed to a simple one P' by a generic and small change in the coefficients of its defining inequalities. This will make non-simple vertices "explode" and become clusters of new vertices, all of which will be simple. This process can not decrease the diameter of the graph, since we can recover the graph of P from that of P' by collapsing certain edges. Hence, to study the Hirsch conjecture, one only needs to consider the simple polytopes:

Lemma 1.3. The diameter of any polytope P is bounded above by the diameter of some simple polytope P' with the same dimension and number of facets.

Graphs of simple polytopes are better behaved than graphs of arbitrary polytopes. Their main property in the context of the Hirsch conjecture is that if u and v are vertices joined by an edge in a simple polytope then there is a single facet containing u and not v, and a single facet containing v and not u. That is, at each step along the graph of P we enter a single facet and leave another one.

Every polytope P (containing the origin in its interior, which can always be assumed by a suitable translation) has a polar polytope P^* whose vertices (respectively facets) correspond to the facets (respectively vertices) of P. More generally, every $(d-i)$-face of P^* corresponds to a face of P of dimension $i-1$, and the incidence relations are reversed.

The polars of simple polytopes are called *simplicial*, and their defining property is that every facet is the convex hull of d points. As an example, the d-dimensional cross polytope is the polar of the d-cube. Since cubes are simple polytopes, cross polytopes are simplicial. The polar of a simplex is a simplex, and simplices are the only polytopes of dimension greater than two which are at the same time simple and simplicial. Since d points spanning an affine $(d-1)$-space are independent, all *facets* (hence all faces) of a simplicial polytope are simplices. This is nice because then we can forget the geometry of P^* and look only at the combinatorics of the simplicial complex formed by its faces. Topologically, that simplicial complex is a sphere of dimension $d-1$.

For simplicial polytopes we can state the Hirsch conjecture as asking how many ridges do we need to cross in order to walk between two arbitrary facets, if we are only allowed to change from one facet to another via a ridge. This suggests defining the *dual graph* $G^{\Delta}(P)$ of a polytope: The undirected graph having as nodes the facets of P and in which two nodes are connected by an edge if and only if their corresponding facets intersect in a ridge of P. In summary, $G^{\Delta}(P) = G(P^*)$.

Why $n-d$? We, and possibly most researchers interested in the question, do not believe the Hirsch conjecture is true. Still, there are several reasons why the bound $n-d$ posed by Hirsch is *natural*:

(1) The conjecture is "invariant" under several standard constructions in polytope theory: products, truncations, and wedges. In particular, using these (and other) constructions, it is possible to obtain polytopes where the bound is met with equality, but not (at least not yet) polytopes that disprove it. We call such polytopes *Hirsch-sharp*. Concerning Hirsch-sharpness our current knowledge, following mostly work of Holt, Fritzsche and Klee [5, 6, 7], is that Hirsch-sharp d-polytopes with n facets:
- Exist if one of the following three conditions holds: $n \le 2d$, $n \le 3d-3$, or $d \ge 7$.
- Do not exist for $d \le 3$ if $n > 2d$, or for $(n,d) \in \{(10,4),(11,4,(12,4)\}$.
- Are unknown but may exist in all other cases: that is, if $d \in \{4,5,6\}$ and $n > 3d-3$, except for the three pairs $(n,4)$ mentioned above.

However, all Hirsch-sharp polytopes with $n > 2d$ that are known are obtained from a single one, by simple geometric operations of wedging, truncating and glueing. That is to say: to some extent, *we only know one non-trivial Hirsch-sharp polytope, the 4-dimensional polytope discovered by Klee and Walkup in their seminal 1967 paper* [15].

(2) The conjecture is equivalent to other two "natural" conjectures: the d-step and non-revisiting conjectures. Here and in the rest of the paper, let $H(n,d)$ denote the maximum diameter of graphs of d-polytopes with n facets.

Theorem 1.4 (Klee-Walkup [15])**.** For any positive integer k, we have $\max_d H(d+k,d) = H(2d,d)$.

As a corollary, the Hirsch conjecture is equivalent to the statement $H(2d,d) = d$ for all d. (The inequality $H(2d,d) \geq d$ follows from the existence of the d-dimensional cubes.) This statement is the *d-step conjecture*. Klee and Walkup also show the equivalence of the Hirsch conjecture to the *non-revisiting conjecture*, which says: *Between any two vertices u and v of a simple polytope there is a path that never revisits a facet that it has previously abandoned.* Such paths are called *non-revisiting* and their length is bounded by $n-d$: at each step we must enter a different facet, and the d facets that our initial vertex lies in cannot be among them.

Theorem 1.5 (Klee-Walkup [15])**.** The Hirsch conjecture, the d-step conjecture, and the non-revisiting conjecture are equivalent.

Positive results. We start by saying for which pairs (n,d) the conjecture is known to be true. The short answer is

(1) $d \leq 3$, and n arbitrary;
(2) $n \leq d+6$ and d arbitrary. This follows from Theorem 1.4 by showing $H(8,4) = 4$, $H(10,5) = 5$ and $H(12,6) = 6$. The first two were proved by Klee and Walkup [15] and the third by Bremner and Schewe [3].
(3) Bremner and Schewe prove also $H(11,4) = 6$. Following up on this work, Bremner, Deza, Hua and Schewe [2] prove $H(12,4) = 7$.

Combining this with the information on Hirsch-sharp polytopes we can give the following "plot" of the function $H(n,d) - (n-d)$ in Table 1. The horizontal coordinate is $n-2d$, so that the column marked "0" corresponds to the polytopes relevant to the d-step conjecture.

$n-2d$	-1	0	1	2	3	4	5	6	7	$\cdots$
d										
2	=	=	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$\cdots$
3	=	=	$<$	$<$	$<$	$<$	$<$	$<$	$<$	$\cdots$
4	=	=	=	$<$	$<$	$<$	?	?	?	$\cdots$
5	=	=	=	=	?	?	?	?	?	$\cdots$
6	=	=	$\geq$	$\geq$	$\geq$	?	?	?	?	$\cdots$
7	=	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\cdots$
8	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\geq$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 1. $H(n,d)$ versus $n-d$, the state of the art

Bounds and special polytopes. In terms of general upper bounds for the diameter of all polytopes, the best bound obtained so far is contained in the paper [9] by G. Kalai and D. J. Kleitman. The proof is extremely simple and elegant (the paper is just two pages!):

Theorem 1.6 (Kalai-Kleitman [9]). $H(n,d) \leq n^2 2^{\log_2 n \log_2 d} = n^{2+\log_2 d}$.

Even if this bound is "quasi-polynomial", its algorithmic implications are not very direct. The proof does not give a clue on how to find a short path towards the vertex maximizing a given functional, or even an explicit path between any pair of vertices. More relevant in the context of the simplex method is the proof that there are "randomized" pivot rules that get to the optimum vertex in *subexponential* time. The exact bound is $e^{K\sqrt{d \log n}}$, where K is a fixed constant [8, 17]. Also interesting are more recent results by Spielman, Teng and Vershynin [19, 21], saying that "random polytopes" have polynomial diameter. More precisely: any polytope can be perturbed to have a diameter that is polynomial in the values of n, d, and the inverse of the perturbation parameter. This result seems to explain why the simplex method works well in practice.

Where the conjectured upper bound can not be proved, it is interesting to study upper bounds for special families of polytopes. Many of the polytopes appearing in combinatorial optimization belong to the class of *network flow polytopes*, which include *transportation polytopes*. Polynomial bounds are known for these and for some generalizations and related classes of polytopes. In addition, polytopes whose vertex coordinates are all zeroes and ones satisfy the Hirsch conjecture.

Negative results. In the second section of the paper we show in detail that the following three natural generalizations of the Hirsch conjecture have been disproved:

(1) The **unbounded** Hirsch conjecture: *Is the diameter of every d-polyhedron P (bounded or not) with n facets at most $n-d$?* This is the conjecture originally posed by Hirsch [4, p. 168]. Klee and Walkup (see [15]) answer this question in the negative, showing there is an unbounded polyhedron of dimension 4, with 8 facets and diameter 5. See Theorem 2.1 for details.

(2) The **monotone** Hirsch conjecture: *Is there, for every d-polytope P with n facets and every linear functional ϕ, a ϕ-monotone path with at most $n-d$ edges from any vertex u to a vertex v where ϕ is maximized? Monotonically* means that we require the value of ϕ to increase at every step. This is relevant since the paths of interest in linear programming are monotone with respect to the functional to be maximized.

Todd's counterexample to this conjecture [20] is a bounded 4-polytope with eight facets and diameter four, in which in order to go monotonically from a certain vertex u to the vertex v where a certain linear functional ϕ is maximized, five steps are needed. See Theorem 2.3.

(3) The **combinatorial** Hirsch conjecture: *If T is a topological triangulation of the $(d-1)$-sphere with n vertices and $G^\Delta(T)$ is its dual graph, is* $\mathrm{diam}(G^\Delta(T))$ *at most $n-d$?* Here, $\mathrm{diam}(G^\Delta(T))$ denotes the diameter of the dual graph, as in the case of simplicial polytopes. This generalizes the Hirsch conjecture since the proper faces of a simplicial d-polytope form a triangulation of the $(d-1)$-sphere. But the converse is not true: starting

in $d = 3$ and with 8 vertices there are *non-polytopal* triangulations of d-spheres, that are not combinatorially isomorphic to the boundary of any polytope.

In [16], Mani and Walkup construct a topological triangulation of the 11-dimensional sphere with 24 vertices whose diameter is more than 12. See Theorem 2.5. Altshuler [1] has shown that the Mani-Walkup sphere is not polytopal.

Even if these three statements have been disproved, it remains to know how far they are from being true. Examples whose diameter is $2(n-d)$ or more would be very significant (current ones achieve $\frac{5}{4}(n-d)$ for the monotone and unbounded conjectures, and $\frac{13}{12}(n-d)$ for the combinatorial one).

2. THREE FALSE GENERALIZATIONS

2.1. **The unbounded and monotone Hirsch conjectures are false.** In the Hirsch conjecture as we have stated it, we only consider bounded polytopes. However, in the context of linear programming the feasible region may well not be bounded, so the conjecture is equally relevant for *unbounded* polyhedra. In fact, that is how W. Hirsch originally posed the question.

Moreover, for the simplex method in linear programming one follows *monotone paths*: starting at an initial vertex u one does pivot steps (that is, one crosses edges) always increasing the value of the linear functional ϕ to be maximized, until one arrives at a vertex v where no pivot step gives a greater value to ϕ. Convexity then implies that v is the global maximum for ϕ in the feasible region. This raises the question whether a *monotone* version of the Hirsch conjecture holds: given two vertices u and v of a polyhedron P and a linear functional that attains its maximum on P at v, is there a ϕ-monotone path of edges from u to v whose length is at most $n-d$?

Here we show that the unbounded and monotone versions of the Hirsch conjecture fail. Both proofs are based on the Hirsch-sharp polytope Q_4 of dimension 4 with nine facets found by Klee and Walkup. In fact, we want to emphasize that knowing the mere existence of such a polytope is enough. We are not going to use any property of Q_4 other than the fact that it is Hirsch-sharp, simple, and has $n > 2d$. Simplicity is not a real restriction since it can always be obtained without decreasing the diameter (Lemma 1.3). The inequality $n > 2d$, however, is essential; the constructions below would not work with, for example, a d-cube.

Theorem 2.1 (Klee-Walkup [15])**.** There is a simple unbounded polyhedron $\tilde{Q}_4$ with eight facets and dimension four and whose graph has diameter 5.

Proof. Let Q_4 be the simple Klee-Walkup polytope with nine facets, and let u and v be vertices of Q_4 at distance five from one another. By simplicity, u and v lie in (at most) eight facets in total and there is (at least) one facet F not containing u nor v. Let $\tilde{Q}_4$ be the unbounded polyhedron obtained by a projective transformation that sends this ninth facet to infinity. The graph of $\tilde{Q}_4$ contains both u and v, and is a subgraph of that of $\tilde{Q}_4$, hence its diameter is still at least five. See Figure 1 for a schematic rendition of this idea. □

Remark 2.2. It is interesting to observe that the "converse" of the above proof also works: from any non-Hirsch unbounded polyhedron $\tilde{Q}$ with eight facets and dimension four, one can build a bounded polytope with nine facets and diameter still five, as follows:

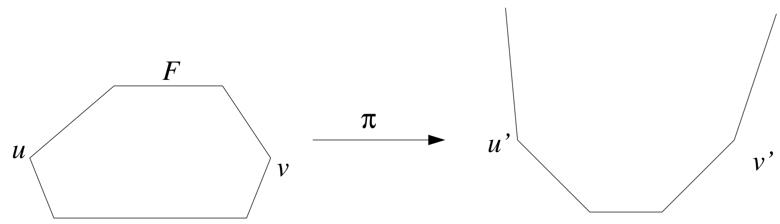

FIGURE 1. Disproving the unbounded Hirsch conjecture

Let u and v be vertices of $\tilde{Q}$ at distance five from one another. Construct the polytope Q by cutting $\tilde{Q}$ with a hyperplane that leaves all the vertices of $\tilde{Q}$ on the same side. This adds a new facet and changes the graph, by adding new vertices and edges on that facet. But u and v will still be at distance five: to go from u to v either we do not use the new facet F that we created (that is, we stay in the graph of $\tilde{Q}_4$) or we use a pivot to enter the facet F and at least another four to enter the four facets containing v: since the Hirsch conjecture holds for 3-dimensional polyhedra, u and v cannot lie in a common facet of $\tilde{Q}$.

We now turn to the monotone version of the Hirsch conjecture:

Theorem 2.3 (Todd [20])**.** There is a simple bounded polytope P, two vertices u and v of it, and a linear functional ϕ such that:

(1) v is the only maximal vertex for ϕ.
(2) Any edge-path from u to v and monotone with respect to ϕ has length at least five.

Proof. Let Q_4 be the Klee-Walkup polytope. Let F be the same "ninth facet" as in the previous proof, one that is not incident to the two vertices u and v that are at distance five from each other. Let H_2 be the supporting hyperplane containing F and let H_1 be any supporting hyperplane at the vertex v. Finally, let H_0 be a hyperplane containing the (codimension two) intersection of H_1 and H_2 and which lies "slightly beyond H_1", as in Figure 2. (Of course, if H_1 and H_2 happen to be parallel, then H_0 is taken to be parallel to them and close to H_1.) The exact condition we need on H_0 is that it does not intersect Q_4 and the small, wedge-shaped region between H_0 and H_1 does not contain the intersection of any 4-tuple of facet-defining hyperplanes of Q_4.

We now make a projective transformation π that sends H_0 to be the hyperplane at infinity. In the polytope $Q_4' = \pi(Q_4)$ we "remove" the facet $F' = \pi(F)$ that is not incident to the two vertices $u' = \pi(u)$ and $v' = \pi(v)$. That is, we consider the polytope Q_4'' obtained from Q_4' by forgetting the inequality that creates the facet F' (see Figure 2 again). Then Q_4'' will have new vertices not present in Q_4', but it also has the following properties:

(1) Q_4'' is bounded. Here we are using the fact that the wedge between H_0 and H_1 contains no intersection of facet-defining hyperplanes: this implies that no facet of Q_4'' can go "past infinity".
(2) It has eight facets: four incident to u' and four incident to v'.
(3) The functional ϕ that is maximized at v' and constant on its supporting hyperplane $H_1' = \pi(H_1)$ is also constant on $H_2' = \pi(H_2)$, and u' lies on the same side of H_1' as v'.

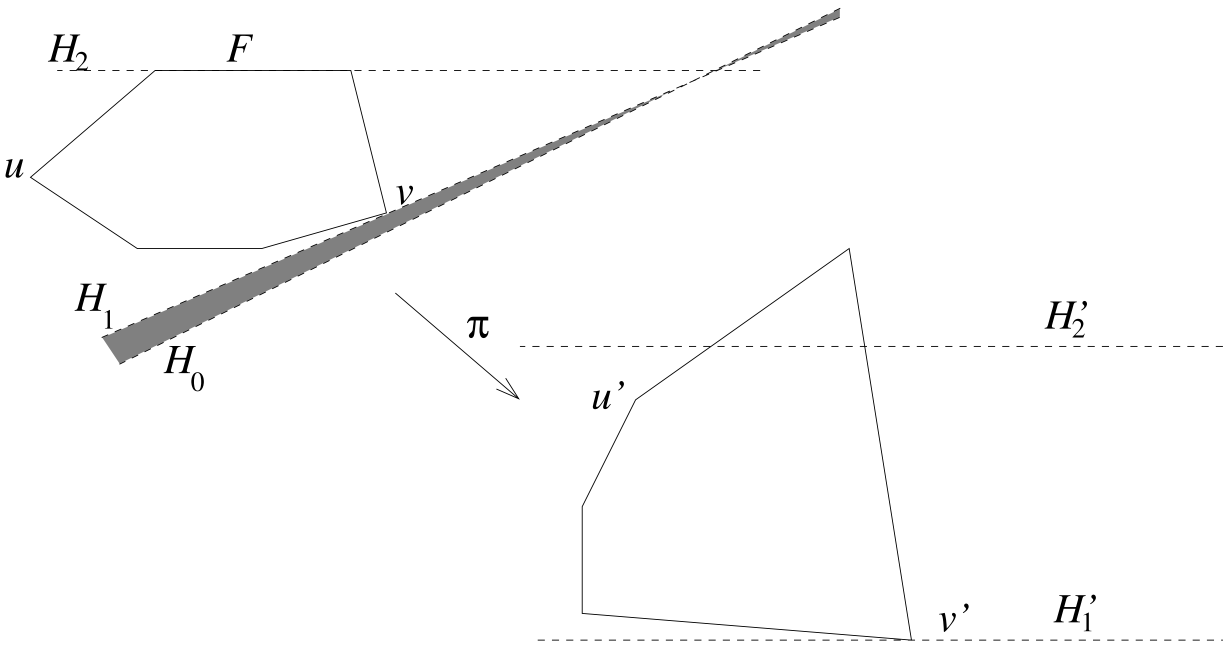

FIGURE 2. Disproving the monotone Hirsch conjecture

In particular, no ϕ-monotone path from u' to v' crosses H_1', which means it is also a path from u' to v' in the polytope Q_4', combinatorially isomorphic to Q_4. □

In both the constructions of Theorems 2.1 and 2.3 one can glue several copies of the initial block Q_4 to one another. We skip details, but in both cases we increase the number of facets by four and the diameter by five, per Q_4 glued, obtaining:

Theorem 2.4 (Klee-Walkup, Todd)**.** (1) There are unbounded 4-polyhedra with $4+4k$ facets and diameter $5k$, for every $k \geq 1$.
(2) There are bounded 4-polyhedra with $5+4k$ facets and vertices u and v of them with the property that any monotone path from u to v with respect to a certain linear functional ϕ maximized at v has length at least $5k$.

This leaves the following open questions:

- Improve these constructions so as to get the ratio "diameter versus facets" bigger than $5/4$. A ratio bigger than two for the unbounded Hirsch conjecture would probably yield counter-examples to the bounded Hirsch conjecture.
- Ziegler [23, p. 87] poses the following *strict* monotone Hirsch conjecture: "for every linear functional ϕ on a d-polytope with n facets there is a ϕ-monotone path of length at most $n-d$". Put differently, in the monotone Hirsch conjecture we add the requirement that not only v but also u has a supporting hyperplane where ϕ is constant.

2.2. **The combinatorial Hirsch conjecture is false.** The other natural generalization of the Hirsch conjecture that we mentioned in the introduction is combinatorial. Since (the boundary of) every simplicial d-polytope is a topological triangulation of the $(d-1)$-dimensional sphere, we can ask whether the simplicial version of the Hirsch conjecture, the one where we walk from simplex to simplex rather than from vertex to vertex, holds for arbitrary triangulations of spheres. The first counterexample to this statement was found by Walkup in 1979 (see [22]), but a simpler one was soon constructed by him and Mani in [16].

Both constructions are based on the equivalence of the Hirsch conjecture to the non-revisiting conjecture (Theorem 1.5). The proof of the equivalence is purely combinatorial, so it holds true for topological spheres. Walkup's initial example is a 4-sphere without the non-revisiting property, and Mani and Walkup's is a 3-sphere:

Theorem 2.5 (Mani-Walkup [16])**.** There is a triangulated 3-sphere with 16 vertices and without the non-revisiting property. Wedging on it eight times one gets a non-Hirsch 11-sphere with 24 vertices.

The part of the Mani-Walkup 3-sphere that implies failure of the non-revisiting property involves only 12 of the 16 vertices. More precisely, Mani and Walkup show the following:

Lemma 2.6. Let K be the three-dimensional simplicial complex on the vertices $a,b,c,d,m,n,o,$ p,q,r,s,t consisting of the following 26 tetrahedra:

abcd		mnop	
abcr	acdr	mnoq	mopq
abdt	bcdt	nops	mnps
abmr	bcnr	anoq	bopq
cdor	dapr	cpmq	dmrq
abmt	bcnt	anos	bops
cdot	dapt	cpms	dmrs

Then:

(1) The complex K can be embedded in a 3-sphere.
(2) No triangulation of the 3-sphere containing K as a subcomplex has the non-revisiting property.
(3) There is a triangulation of the 3-sphere with 16 vertices and containing K as a subcomplex.

We are not going to prove part 3 of the lemma. The construction is somehow complicated and, moreover, in a sense that part is irrelevant. Indeed, once we know that K can be embedded in a 3-sphere we can rely on general topological results to conclude that K can be completed to a triangulation of the whole 3-sphere. The only drawback of this approach is that we cannot control *a priori* the number of extra vertices needed in the completion, but that will only affect the number of vertices of the final 3-sphere (and the number of suspensions needed to get a non-Hirsch sphere from it).

Proof of parts 1 and 2 of Lemma 2.6. The proof follows from the following description of the simplicial complex K: it consists of two triangulations of bipyramids over the octagons $a-m-b-n-c-o-d-p$ and $a-o-b-p-c-m-d-n$, glued along the eight vertices of the octagons. See Figure 3.

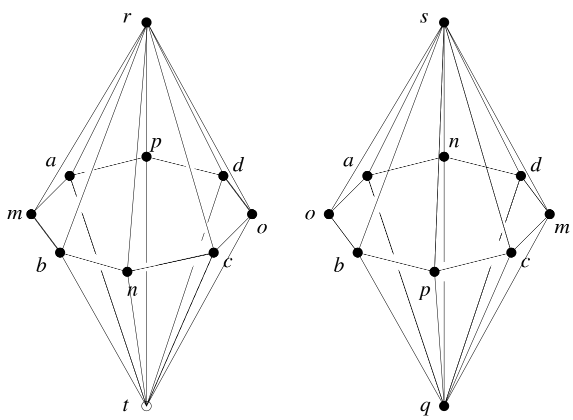

FIGURE 3. Two octagonal bipyramids in the triangulation of S depicting the outside-most 16 tetrahedra

Once this is shown, part 1 is easy. Since we are living in a topological world, we can "pinch" the equatorial vertices of one of the octagons and there is no obstruction to glue them to their counterparts in the other octagon. One key property is that we are not glueing any of the edges: the order of vertices in the two octagons is not the same, and it is designed so that no edge appears in both.

For part 2, let us see the construction in more detail. It starts with a core tetrahedron inside each bipyramid, namely *abcd* and *mnop*. See Figure 4.

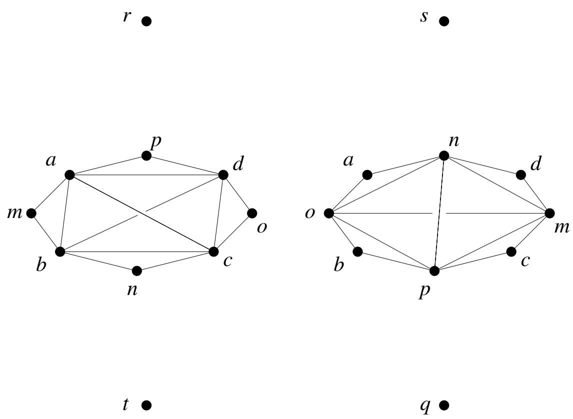

FIGURE 4. The simplices *abcd* and *mnop*

Each of these tetrahedra is surrounded by two tetrahedra joined to each apex of the corresponding bipyramid, as shown in Figure 5. These are the tetrahedra in the second and third line of the statement and together with the initial ones they triangulate two octahedra. Finally, these octahedra are each surrounded by eight more tetrahedra each: those obtained joining the four triangles left uncovered in each octagon (see Figure 5 again) to the two apices of their bipyramid.

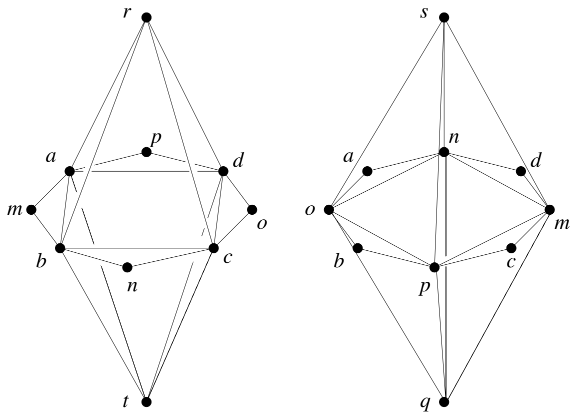

FIGURE 5. Eight additional simplices in the Mani-Walkup triangulation

From this description it is easy to prove part 2 of the lemma, as follows: Every path from the tetrahedron *abcd* to the tetrahedron *mnop* must leave the bipyramid on the left of Figure 3, and it will do so through one of the sixteen boundary triangles. These triangles are the joins of the eight edges of the octagon to the two apices. In particular, our path will at this point have abandoned three of the vertices of *abcd* and entered one of *mnop*. For the non-revisiting property to hold, the abandoned ones should not be used again, and the entered one should not be abandoned, since it is a vertex of our target tetrahedron. But then it is impossible for us to enter the second bipyramid: we should do so via another triangle that joins an octagon edge to an apex, and non-revisiting implies that this edge should use the same vertex form *abcd* and the same vertex from *mnop*. This is impossible since the two octagons have no edge in common.

Let us explain this in a concrete example. By symmetry, there is no loss of generality in assuming that we exit from the left bipyramid via the triangle *amr*. Since we cannot abandon *m*, we must enter the second bipyramid via one of the boundary triangles using *m*, namely one of *mcs*, *mcq*, *mds* or *mdq*. This violates the non-revisiting property, since *c* and *d* had already been abandoned. □

Unfortunately, this triangulated 3-sphere does not give a counterexample to the Hirsch conjecture. It would give a counterexample if it were *polytopal*. That is, if it were combinatorially isomorphic to the boundary complex of a four-dimensional simplicial polytope. However, Altshuler [1] has shown that (for the explicit completion of the subcomplex K given in [16]) this is not the case. As far as we know it remains an open question to show that *no completion of K* to the 3-sphere is polytopal, but we believe that to be the case. Even more strongly, we believe that K cannot be embedded in $\mathbb{R}^3$ with linear tetrahedra, a necessary condition for polytopality by the well-known Schlegel construction [23].

As in the monotone and bounded cases, several copies of the construction can be glued to one another. Doing so provides triangulations of the 11-sphere with $12+12k$ vertices and diameter at least $13k$, for any k.

REFERENCES

[1] A. Altshuler. The Mani-Walkup spherical counterexamples to the W_v-path conjecture are not polytopal. *Math. Oper. Res.*, 10(1):158–159, 1985.

[2] D. Bremner, A. Deza, W. Hua, and L. Schewe. More bounds on the diameters of convex polytopes, preprint 2009. `http://arxiv.org/abs/0911.4982`

[3] D. Bremner and L. Schewe. Edge-graph diameter bounds for convex polytopes with few facets, preprint 2008. `http://arxiv.org/abs/0809.0915`

[4] G. B. Dantzig, Linear programming and extensions, Princeton University Press, 1963.

[5] K. Fritzsche and F. B. Holt. More polytopes meeting the conjectured hirsch bound. *Discrete Math.*, 205:77–84, 1999.

[6] F. B. Holt. Blending simple polytopes at faces. *Discrete Math.*, 285:141–150, 2004.

[7] F. Holt and V. Klee. Many polytopes meeting the conjectured Hirsch bound. *Discrete Comput. Geom.*, 20:1–17, 1998.

[8] G. Kalai. A subexponential randomized simplex algorithm. In *Proceedings of the 24th annual ACM symposium on the Theory of Computing*, pages 475–482, Victoria, 1992. ACM Press.

[9] G. Kalai and D. J. Kleitman. A quasi-polynomial bound for the diameter of graphs of polyhedra. *Bull. Amer. Math. Soc.*, 26:315–316, 1992.

[10] N. Karmarkar. A new polynomial time algorithm for linear programming. *Combinatorica*, 4(4):373–395, 1984.

[11] L. G. Hačijan. A polynomial algorithm in linear programming. (in Russian) *Dokl. Akad. Nauk SSSR*, 244(5):1093–1096, 1979.

[12] E. D. Kim, and F. Santos. An update on the Hirsch conjecture, preprint 2009. `http://arxiv.org/abs/0907.1186v2`.

[13] V. Klee, G. J. Minty, How good is the simplex algorithm?, in *Inequalities, III (Proc. Third Sympos., Univ. California, Los Angeles, Calif., 1969; dedicated to the memory of Theodore S. Motzkin)*, Academic Press, New York, 1972, pp. 159–175.

[14] V. Klee, P. Kleinschmidt, The d-Step Conjecture and Its Relatives, *Mathematics of Operations Research*, 12(4):718–755, 1987.

[15] V. Klee and D. W. Walkup. The d-step conjecture for polyhedra of dimension $d<6$. *Acta Math.*, 133:53–78, 1967.

[16] P. Mani and D. W. Walkup. A 3-sphere counterexample to the W_v-path conjecture. *Math. Oper. Res.*, 5(4):595–598, 1980.

[17] J. Matoušek, M. Sharir, and E. Welzl. A subexponential bound for linear programming. In *Proceedings of the 8th annual symposium on Computational Geometry*, pages 1–8, 1992.

[18] S. Smale, Mathematical problems for the next century. *Mathematics: frontiers and perspectives*, pp. 271–294, American Mathematics Society, Providence, RI (2000).

[19] D. A. Spielman and S. Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. *J. ACM*, 51(3):385–463, 2004.

[20] M. J. Todd. The monotonic bounded Hirsch conjecture is false for dimension at least 4, *Math. Oper. Res.*, 5:4, 599–601, 1980.

[21] R. Vershynin. Beyond Hirsch conjecture: walks on random polytopes and smoothed complexity of the simplex method. In *IEEE Symposium on Foundations of Computer Science*, volume 47, pages 133–142. IEEE, 2006.

[22] D. W. Walkup. The Hirsch conjecture fails for triangulated 27-spheres. *Math. Oper. Res.*, 3:224-230, 1978.

[23] G. M. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, 152, Springer-Verlag, 1995.

A Short Introduction to Algebraic Tropical Geometry

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Tropical geometry is a growing field of mathematics that has received a lot of attention in recent years. There are various different approaches and applications of tropical geometry which establish a deep connection between algebraic geometry and combinatorics; see [5, 7] for general overviews.

From an algebraic point of view tropical geometry provides a tool to investigate algebraic varieties by associating certain combinatorial objects to them. This is done by considering the image of an algebraic variety X under a valuation map; see [3, 8, 11]. The set of real-valued points of this image has the structure of a polyhedral complex in $\mathbb{R}^n$. This object is defined to be the tropical variety of X and can be used to obtain information on the original variety as is done for example in [3]. For practical purposes there is a useful characterization of tropical varieties in terms of initial polynomials given in [11], fully proved in [3, Theorem 4.2] and more explicitly in [8]. From this it follows that in the case of constant coefficients, i.e. if the valuation on the ground field is trivial, the tropical variety of a projective algebraic variety is a subfan of the Gröbner fan of its defining ideal. It contains exactly those cones of the Gröbner fan corresponding to initial ideals that do not contain a monomial.

We will only consider this constant coefficient case and we will define the tropical variety as a fan associated to an ideal I instead of a projective variety. In this situation the ideal I need not be a radical ideal. So for our purposes let K be an infinite field and $K[x_1,\dots,x_n]$ be the polynomial ring in n variables over K. In this setting the tropical variety $T(I)$ of a graded ideal $I \subset K[x_1,\dots,x_n]$ is defined as the fan

$$T(I) = \{\omega \in \mathbb{R}^n : \mathrm{in}_\omega(I) \text{ does not contain a monomial}\},$$

where the fan structure is the one induced by the Gröbner fan of I.

In order to be able to obtain information on I from its tropical variety, it is a natural question to ask which algebraic invariants of I can be seen in $T(I)$. One example where this is possible is the result following from [1] that the Krull dimension of I equals the dimension of $T(I)$ as a polyhedral complex, whenever I does not contain a monomial. However, it is not always possible for $T(I)$ to contain information on I. If I contains a monomial, then $T(I) = \emptyset$ and no information can be recovered. But even if $T(I) \neq \emptyset$ we cannot, for example, always recover the depth of I; see [10]. One reason for this is that the tropical variety depends on the choice of coordinates of I in the following sense.

For any $g = (g_{ij}) \in \mathrm{GL}_n(K)$ we consider the K-algebra automorphism on $K[x_1,\dots,x_n]$ induced by

$$\begin{aligned} g : K[x_1,\dots,x_n] &\longrightarrow K[x_1,\dots,x_n] \\ x_i &\longmapsto \sum_{j=1}^{n} g_{ij}x_j. \end{aligned}$$

We identify g with the induced automorphism and use the notation g for both of them. Note that for any $g \in \mathrm{GL}_n(K)$ the ideal $g(I)$ is a graded ideal with the same algebraic properties as I. On the other hand in general we have $T(I) \neq T(g(I))$ for $g \in \mathrm{GL}_n(K)$. For example for $I = (x_1) \subset K[x_1,x_2]$ we have $T(I) = \emptyset$, but $T(g(I)) = T((x_1 + x_2)) = \mathbb{R}(1,1)$ for $g \in \mathrm{GL}_2(K)$ with $g_{11} = g_{12} = 1$. Analogous to the existence of a generic initial ideal in Gröbner bases theory (see [4, 6]) we can show, however, that for a generic choice of coordinates the Gröbner fan and the tropical variety are both constant.

Theorem. [9, Theorem 3.1, Corollary 6.9] Let $I \subset K[x_1,\dots,x_n]$ be a graded ideal. There exists a Zariski-open subset $\emptyset \neq U \subset \mathrm{GL}_n(K)$ such that:

(1) the Gröbner fan of $g(I)$ is the same fan for all $g \in U$.
(2) the tropical variety $T(g(I))$ is the same fan for all $g \in U$.

Since a non-empty Zariski-open set is dense in $\mathrm{GL}_n(K)$, the fans fulfilling the conditions of this theorem are unique. They will be called the generic Gröbner fan of I and the generic tropical variety $\mathrm{gT}(I)$ of I respectively. In particular this yields a natural way to associate a non-empty tropical variety even to ideals which contain a monomial. In the example $I = (x_1) \subset K[x_1,x_2]$ we have $g(I) = (g_{11}x_1 + g_{12}x_2)$ for $g \in \mathrm{GL}_2(K)$. If $g_{11}, g_{12} \neq 0$, it follows that $T(g(I)) = \mathbb{R}(1,1)$. But the set of all $g \in \mathrm{GL}_2(K)$ fulfilling this condition is Zariski-open, as its complement is defined by the polynomials $g_{11} = 0, g_{12} = 0$. Hence, in this case the generic tropical variety of I is $\mathrm{gT}(I) = \mathbb{R}(1,1)$.

The generic tropical variety of an ideal relates to some important invariants of the ideal in a direct way. To see this we can first describe the underlying set of the generic tropical variety. This set is always a skeleton of the normal fan of an $n-1$-simplex in $\mathbb{R}^n$ and just depends on the dimension of the ideal. More precisely let $\mathscr{W}_n$ be the complete fan consisting of the closed cones $C_A = \left\{\omega \in \mathbb{R}^n : \omega_i = \min_j \left\{\omega_j\right\} \text{ for all } i \in A\right\}$ for every set $\emptyset \neq A \subset \{1,\dots,n\}$ and for $0 \leq m \leq n$ let $\mathscr{W}_n^m$ be the m-skeleton of $\mathscr{W}_n$. Then we have:

Theorem. [9, Corollary 8.4] Let $I \subset K[x_1,\dots,x_n]$ be a graded ideal of Krull dimension $m < n$. Then as a set $\mathrm{gT}(I) = \mathscr{W}_n^m$.

This shows that generically the Krull dimension of I is the only algebraic invariant of I which can be obtained from the tropical variety as a set.

For special classes of ideals the generic tropical variety can be computed directly. In particular this is true for linear ideals. In this case the generic tropical variety also inherits the fan structure from $\mathscr{W}_n$. Moreover, we can also compute the generic Gröbner fan for linear ideals. It turns out that the generic Gröbner fan of an m-dimensional linear ideal I has a lot more m-dimensional cones than its generic tropical variety as observed in [10]. This gives a negative answer to the question, whether $\mathrm{gT}(I)$ is exactly the m-skeleton of the generic Gröbner fan of I.

In general to obtain more information on I than its dimension it is necessary to look at the fan structure on $\mathrm{gT}(I)$ induced by the Gröbner fan. If we know the depth of I to be greater than 0, it is possible to recover it from the fan $\mathrm{gT}(I)$ up to the almost-Cohen-Macaulay case. Here I is said to be almost-Cohen-Macaulay if $\mathrm{depth}(I) = \dim I - 1$. More precisely we can state:

Theorem. [10, Theorem 4.9] Let $I \subset K[x_1,\dots,x_n]$ be a graded ideal with $\dim I = m > 0$ and $\mathrm{depth}(I) > 0$. Then I is Cohen-Macaulay or almost-Cohen-Macaulay if and only if $\mathrm{gT}(I) = \mathscr{W}_n^m$ as a fan.

The underlying reason for the relationship between the algebraic property $\mathrm{depth}(I)$ and the fan structure of $\mathrm{gT}(I)$ is given by the fact that the depth of I can be read off from its generic initial ideal with respect to the revlex order. Since the generic initial ideals of I correspond to the full dimensional cones of its generic Gröbner fan, this invariant determines to a certain degree how refined the fan structure of $\mathrm{gT}(I)$ is.

In addition to the depth one we also obtain information on the Hilbert-Samuel multiplicity of I if we consider $\mathrm{gT}(I)$ as a weighted fan. There is a natural way of associating weights (called multiplicities) to the maximal cones of a tropical variety as done in [2]. These weights do not always reflect information on the multiplicity of I as stated in [10]. In the generic case however there is a direct correspondence between the two notions of multiplicity:

Theorem. [10, Theorem 6.8] Let $I \subset K[x_1,\dots,x_n]$ be a graded ideal with $\dim I > 0$. Then the intrinsic multiplicity in the sense of [2] of every maximal cone of $\mathrm{gT}(I)$ is equal to the Hilbert-Samuel multiplicity of I.

Hence, the hope that the tropical variety is a simpler object than the original variety which nevertheless recalls some information fulfills itself well in a generic setting in the constant coefficient case. Here we can see that the algebraic invariants dimension and depth of I show up directly in the generic tropical variety and that the multiplicity of I corresponds to a natural definition of multiplicities on the maximal cones of $\mathrm{gT}(I)$.

REFERENCES

[1] R. Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*. J. Reine Angew. Math. **347**, 168–195 (1984).

[2] A. Dickenstein, E. M. Feichtner and B. Sturmfels, *Tropical discriminants*. Journal of the American Mathematical Society **20**, 1111–1133 (2007).

[3] J. Draisma, *A tropical approach to secant dimensions*. J. Pure Appl. Algebra **212**, No. 2, 349–363 (2008).

[4] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics **150**, Springer (1995).

[5] A. Gathmann, *Tropical algebraic geometry*. Jahresber. Dtsch. Math.-Ver. **108**, No. 1, 3–32 (2006).

[6] M.L. Green, *Generic initial ideals*. In: J. Elias (ed.) et al., Six lectures on commutative algebra. Birkhäuser, Prog. Math. **166**, 119–186 (1998).

[7] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical algebraic geometry*. Oberwolfach Seminars **35**, Birkhäuser (2007).

[8] A. Nedergaard Jensen, H. Markwig and T. Markwig, *An algorithm for lifting points in a tropical variety*. Collect. Math. **59**, No. 2, 129–165 (2008).

[9] T. Römer, K. Schmitz, *Generic Tropical Varieties.* arXiv:0904.0120.

[10] T. Römer, K. Schmitz, *Algebraic Properties of Generic Tropical Varieties.* arXiv:0909.2129.

[11] D. Speyer and B. Sturmfels, *The tropical Grassmannian*. Adv. Geom. **4**, No. 3, 389–411 (2004).

Algebraic versus topological triangulated categories

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Introduction. The most commonly known triangulated categories arise from chain complexes in an abelian category by passing to chain homotopy classes or inverting quasi-isomorphisms. Such examples are called 'algebraic' because they originate from abelian (or at least additive) categories. Stable homotopy theory produces examples of triangulated categories by quite different means, and in this context the source categories are usually very 'non-additive' before passing to homotopy classes of morphisms. Because of their origin I refer to these examples as 'topological triangulated categories'.

There are systematic differences between these two kinds of triangulated categories: certain properties – defined entirely in terms of the triangulated structure – hold in all algebraic examples, but fail in some topological ones. These differences are all torsion phenomena, and rationally there is no difference between algebraic and topological triangulated categories.

This is an extended abstract of a talk given at the workshop *Combinatorial Structures in Algebra and Topology* in Osnabrück. A more detailed exposition can be found in [13].

Algebraic and topological triangulated categories. A triangulated category is *algebraic* in the sense of Keller [7, 3.6] if it is triangle equivalent to the stable category of a Frobenius category, i.e., an exact category with enough injectives and enough projectives in which injectives and projectives coincide. Examples include all triangulated categories which one should reasonably think of as 'algebraic': various homotopy categories and derived categories of rings, schemes and abelian categories; stable module categories of Frobenius rings; derived categories of modules over differential graded algebras and differential graded categories. By a theorem of Porta [11], every algebraic triangulated category which is *well generated* (a mild restriction on its 'size') is equivalent to a localization of the derived category $\mathscr{D}(\mathscr{A})$ of a small differential graded category $\mathscr{A}$.

A triangulated category is *topological* if it is equivalent to a full triangulated subcategory of the homotopy category of a stable model category. A *model category* (in the sense of Quillen [12]) is an axiomatic framework for homotopy theoretic constructions; a model category is *stable* if it has a zero object and the suspension functor is a self-equivalence of its homotopy category. The homotopy category of a stable model category is naturally triangulated, cf. [5, 7.1.6].

An important example of a topological triangulated category is the *stable homotopy category* of algebraic topology which was first introduced by Boardman (unpublished; accounts of Boardman's construction appear in [16] and [1, Part III]). There is an abundance of models for the stable homotopy category, see for example [2, 3, 6, 8]. A related example is the Spanier-Whitehead category which can be obtained from the homotopy category of finite based CW-complexes by formally inverting the suspension functor (see [15] or [9, Ch. 1, §2]). The Spanier-Whitehead category predates the stable homotopy category and is triangulated equivalent to the full subcategory of *compact* (or finite) objects in the stable homotopy category. Further examples of topological triangulated categories are 'derived' (i.e., homotopy) categories of structured ring spectra, equivariant and motivic stable homotopy categories, sheaves of spectra on a Grothendieck site or localizations of all these, see [14, Sec. 2.3] for more details. In [4, Thm. 4.7] Heider essentially shows that every well generated topological triangulated category is equivalent to a localization of the homotopy category $\mathrm{Ho}(\mathscr{R}\text{-mod})$ of a small spectral category $\mathscr{R}$.

Algebraic triangulated categories are typically also topological – the converse is not generally true, and that is the point of this talk. Examples of triangulated categories which are neither algebraic nor topological were recently constructed by Muro, Strickland and the author [10]. The simplest one is the category $\mathscr{F}(\mathbb{Z}/4)$ of finitely generated free modules over the ring $\mathbb{Z}/4$. The category $\mathscr{F}(\mathbb{Z}/4)$ has a unique triangulation with the identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact.

The n-order. For an object X of a triangulated category $\mathscr{T}$ and a natural number n we write $n \cdot X$ for the n-fold multiple of the identity morphism in the group $[X,X]$ of endomorphisms in $\mathscr{T}$. We let X/n denote any cone of $n \cdot X$, i.e., any object which is part of a distinguished triangle

$$X \xrightarrow{n\cdot} X \longrightarrow X/n \longrightarrow X[1] .$$

A short diagram chase shows that the group $[X/n, X/n]$ is always annihilated by n^2; in algebraic triangulated categories, more is true: in the world of chain complexes, one can write down an explicit nullhomotopy of of $n \cdot X/n$. In general, however, we can have $2 \cdot X/2 \neq 0$ for certain objects X; the free module of rank one in the above triangulated category $\mathscr{F}(\mathbb{Z}/4)$ or the mod-2 Moore spectrum in the stable homotopy category are two examples. This shows in particular that $\mathscr{F}(\mathbb{Z}/4)$ and the stable homotopy category are not algebraic.

In topological triangulated categories, the phenomenon that we can have $n \cdot X/n \neq 0$ is entirely 2-local. More precisely, if $\mathscr{T}$ is a topological triangulated category and p an odd prime, then $p \cdot X/p = 0$ for every object X of $\mathscr{T}$. The reason behind this fact is that a topological triangulated category admits an action of the stable homotopy category, an object X/n is isomorphic to the product of X with the mod-n Moore spectrum and for odd primes p, the mod-p Moore spectrum is annihilated by p (which is not the case for $p = 2$).

From the above, it is still conceivable that every topological triangulated category in which 2 is invertible is algebraic. In order to distinguish algebraic from topological triangulated categories away from the prime 2 we need a more refined invariant, the *n-order* of an object in a triangulated category. As before we denote by K/n any cone of $n \cdot K$, which comes as part of a distinguished triangle

$$K \xrightarrow{n\cdot} K \xrightarrow{\pi} K/n \longrightarrow K[1] .$$

An *extension* of a morphism $f : K \longrightarrow Y$ is then a morphism $\bar{f} : K/n \longrightarrow Y$ satisfying $\bar{f}\pi = f$. Such an extension exists if and only if $n \cdot f = 0$, and then the extension will usually not be unique.

Definition. Consider a triangulated category $\mathscr{T}$ and a natural number $n \geq 2$. We define the *n-order* for objects Y of $\mathscr{T}$ inductively.

- Every object has n-order greater or equal to 0.
- For $k \geq 1$, an object Y has n-order greater or equal to k if and only if for every object K of $\mathscr{T}$ and every morphism $f : K \longrightarrow Y$ there exists an extension $\bar{f} : K/n \longrightarrow Y$ such that some (hence any) mapping cone of $\bar{f}$ has n-order greater or equal to $k-1$.

The n-order of Y is then the largest k (possibly infinite) such that Y has n-order greater or equal to k. The n-order of the triangulated category $\mathscr{T}$ as the n-order of some (hence any) zero object. The following properties are direct consequences of the definitions.

- The n-order for objects is invariant under isomorphism and shift.
- An object Y has positive n-order if and only if every morphism $f : K \longrightarrow Y$ has an extension to K/n, which is equivalent to $n \cdot f = 0$. So $n\text{-ord}(Y) \geq 1$ is equivalent to the condition $n \cdot Y = 0$.
- The n-order of a triangulated category is one larger than the minimum of the n-orders of all objects of the form K/n.
- For a full triangulated subcategory $\mathscr{S} \subseteq \mathscr{T}$ and any object Y of $\mathscr{S}$ we have $n\text{-ord}^{\mathscr{S}}(Y) \geq n\text{-ord}^{\mathscr{T}}(Y)$. In particular $n\text{-ord}(\mathscr{S}) \geq n\text{-ord}(\mathscr{T})$.
- If $\mathscr{T}$ is a $\mathbb{Z}[1/n]$-linear triangulated category, then K/n is trivial for every object K and thus $\mathscr{T}$ has infinite n-order. If on the other hand Y is non-trivial, then $n\text{-ord}(Y) = 0$.
- If every object of $\mathscr{T}$ has positive n-order, then $n \cdot Y = 0$ for all objects Y and so $\mathscr{T}$ is a $\mathbb{Z}/n$-linear triangulated category. If conversely $\mathscr{T}$ is a $\mathbb{Z}/n$-linear triangulated category, then every object has infinite n-order.

The last two items show that the n-order is useless if $\mathscr{T}$ is a *k-linear* triangulated category for some field k, since then every $n \in \mathbb{Z}$ is either zero or a unit in k.

Our main results about the n-order in triangulated categories are as follows.

Theorem. *Let $\mathscr{T}$ be an algebraic triangulated category and X an object of $\mathscr{T}$. Then the object X/n has infinite n-order. In particular, every algebraic triangulated category $\mathscr{T}$ has infinite n-order.*

Theorem. *Let $\mathscr{T}$ be a topological triangulated category and X an object of $\mathscr{T}$. Then for any prime p, the object X/p has p-order at least $p-2$. In particular, every topological triangulated category $\mathscr{T}$ has p-order at least $p-1$.*

Theorem. *Let p be a prime. Then in the Spanier-Whitehead category, the mod-p Moore spectrum S/p has p-order $p-2$. Hence the Spanier-Whitehead category has p-order $p-1$.*

There are some interesting open questions left. For example, can we have $p \cdot X/p \neq 0$ for an odd prime p? More generally, is there an odd prime p and a triangulated category whose p-order is strictly less than $p-1$? Examples would necessarily have to be 'exotic', i.e., arise in triangulated categories which are neither algebraic nor topological. One should beware that the obvious analog $\mathscr{F}(\mathbb{Z}/p^2)$ of the triangulated category $\mathscr{F}(\mathbb{Z}/4)$ is not triangulated (already the rotation axiom fails since p and $-p$ are different modulo p^2 when p is odd).

Our invariant to distinguish the different kinds of triangulated categories is based on torsion phenomena; the n-order is useless rationally since $\mathbb{Q}$-linear triangulated categories have infinite

n-order for all n. In fact it turns out that rationally the notions of algebraic and topological triangulated categories essentially coincide, i.e., at least under mild technical assumptions and cardinality restriction, every $\mathbb{Q}$-linear topological triangulated category is algebraic. At present, I do not know of a $\mathbb{Q}$-linear triangulated category which is not topological.

REFERENCES

[1] J. F. Adams, *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, 1974. x+373 pp.

[2] A. K. Bousfield, E. M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets.* Geometric applications of homotopy theory II. Lecture Notes in Math., vol. 658, Springer, 1978, pp. 80–130.

[3] A. D. Elmendorf, I. Kriz, M. A. Mandell, J. P. May, *Rings, modules, and algebras in stable homotopy theory. Appendix by M. Cole*, Mathematical Surveys and Monographs **47**, Amer. Math. Soc. 1997, xii+249 pp.

[4] A. Heider, *Two results from Morita theory of stable model categories.* `arXiv:0707.0707`

[5] M. Hovey, *Model categories.* Mathematical Surveys and Monographs, **63**, Amer. Math. Soc. 1999, xii+209 pp.

[6] M. Hovey, B. Shipley, J. Smith, *Symmetric spectra.* J. Amer. Math. Soc. **13** (2000), 149–208.

[7] B. Keller, *On differential graded categories*. Proceedings of the International Congress of Mathematicians, Madrid, Spain, 2006, vol. II, European Mathematical Society, 2006, pp. 151–190.

[8] M. A. Mandell, J. P. May, S. Schwede, B. Shipley, *Model categories of diagram spectra.* Proc. London Math. Soc. **82** (2001), 441–512.

[9] H. R. Margolis, *Spectra and the Steenrod algebra. Modules over the Steenrod algebra and the stable homotopy category*, North-Holland Mathematical Library **29**, North-Holland Publishing Co., 1983, xix+489 pp.

[10] F. Muro, S. Schwede, N. Strickland, *Triangulated categories without models.* Invent. Math. **170** (2007), 231-241.

[11] M. Porta, *The Popescu-Gabriel theorem for triangulated categories.* `arXiv:0706.4458`

[12] D. G. Quillen, *Homotopical algebra.* Lecture Notes in Math. **43**, Springer-Verlag, 1967.

[13] S. Schwede, *Algebraic versus topological triangulated categories.* To appear in: 'Triangulated categories', Cambridge Univ. Press. Editors T. Holm, P. Jorgensen, R. Rouquier,

[14] S. Schwede, B. Shipley, *Stable model categories are categories of modules.* Topology **42** (2003), no. 1, 103–153.

[15] E. Spanier, J. H. C. Whitehead, *A first approximation to homotopy theory.* Proc. Nat. Acad. Sci. USA **39** (1953), 655-660.

[16] R. Vogt, *Boardman's stable homotopy category.* Lecture Notes Series, No. 21 Matematisk Institut, Aarhus Universitet 1970 i+246 pp.

Motivic invariants of algebraic groups

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Let G denote a simple linear algebraic group defined over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ is a compact Lie group. It is easy to see that such G exists, and the Hasse principle for algebraic groups tells that this group is unique.

In 1999 J.-P. Serre posed the following problem:

Let K be a field of characteristic 0. Is it true that G_K is a split group if and only if -1 is a sum of 16 squares of the field K?

In the same year M. Rost proved the "if" part. A positive solution of Serre's problem would imply several results about the structure of finite subgroups of algebraic groups of type E_8 over *arbitrary fields*. E.g.,

Corollary 0.1 ([GS09]). *Let G be a split group of type E_8 over a field K of characteristic 0. Then $\mathrm{PGL}(2,31)$ is a subgroup of $G(K)$ iff -1 is a sum of 16 squares in K, and $\mathrm{SL}(2,32)$ is a subgroup of $G(K)$ iff -1 is a sum of 16 squares in K and $\cos(2\pi/11)$ is in K.*

Any group G of type E_8 over any field K can be viewed as an element in the Galois cohomology $H^1(K,G_0)$, where G_0 is the split group of type E_8. Therefore one may speak about the *Rost invariant* of G (see [KMRT]). Moreover, one can show that the Rost invariant of the compact Lie group of type E_8 is zero. Therefore Serre's problem follows from the following more general result:

Theorem 0.2 ([Se08]). *Let G be a group of type E_8 over a field K of characteristic 0 with trivial Rost invariant. Then there exists a functorial element $u \in H^5(K,\mathbb{Z}/2)$ such that for any field extension L/K we have $u_L = 0$ iff G_L splits over an odd degree extension of L.*

For the compact group G the invariant $u = (-1)^5$. In particular, the invariant u is not always zero. Observe that this theorem implies that any group of type E_8 over a field of cohomological dimension 2 and characteristic 0 splits over an odd degree field extension. This is a weaker version of the Serre Conjecture II.

Theorem 0.2 follows from the following two theorems:

Theorem 0.3 ([Se08]). *Let X be a smooth projective variety over a field K of characteristic 0 with no zero-cycles of odd degree. Assume that the Chow motive of X with $\mathbb{Z}/2$-coefficients has a direct summand R such that $X \otimes R \simeq X \oplus X\{\dim X\}$ in the category of Chow motives. Then $\dim X = 2^{n-1} - 1$ for some n, and there exists a functorial element $u \in H^n(K,\mathbb{Z}/2)$ such that for any field extension L/K we have $u_L = 0$ iff X_L has a zero-cycle of odd degree.*

Theorem 0.4 ([Se08]). *Let G be a group of type E_8 over a field K with trivial Rost invariant. Then the variety of Borel subgroups of G has a direct summand R satisfying conditions of Theorem 0.3*

The proof of Theorem 0.3 heavily uses the motivic technique of Voevodsky and results of Rost (see [OVV07], [Ro07], and [Vo98]). There is a wide generalization of Theorem 0.4 to arbitrary linear algebraic groups. The motives R which one can associate with any semisimple algebraic group, can be described in terms of an invariant, called the J-invariant (see [PSZ08] and [PS09]). Together they are motivic invariants from the title.

REFERENCES

[GS09] S. Garibaldi, N. Semenov, *Degree* 5 *invariant of* E_8, Preprint 2009, available from `http://arxiv.org`

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, J.-P. Tignol, *The book of involutions*, AMS Colloquium Publ., 44, AMS, Providence, RI, 1998.

[OVV07] D. Orlov, A. Vishik, V. Voevodsky, *An exact sequence for* $K_*^M/2$ *with applications to quadratic forms*, Ann. Math. **165** (2007), no. 1, 1–13.

[PSZ08] V. Petrov, N. Semenov, K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Sci. Éc. Norm. Sup. **41** (2008), 1023–1053.

[PS09] V. Petrov, N. Semenov, *Generically split projective homogeneous varieties*, 2009, to appear in Duke Math. J.

[Ro07] M. Rost, *On the basic correspondence of a splitting variety*, Preprint 2007, available from `http://www.math.uni-bielefeld.de/~rost`

[Se08] N. Semenov, *Motivic construction of cohomological invariants*, Preprint 2008, available from `http://arxiv.org`

[Vo98] V. Voevodsky, *Motivic cohomology with* $\mathbb{Z}/2$*-coefficients*, Publ. Math. de l'IHÉS **98** (2003), 59–104.

Hilbert Depth and Positivity

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1. INTRODUCTION

Let $\mathbb{F}$ be an infinite field. We consider the ring $R = \mathbb{F}[X_1, \dots, X_n]$, equipped with the standard $\mathbb{Z}$–grading, i. e., $\deg(X_i) = 1$ for $i = 1, \dots, n$. Furthermore let $M \neq 0$ be a finitely generated $\mathbb{Z}$–graded R–module.

Every homogenous component of M is a finite–dimensional $\mathbb{F}$–vectorspace, and since R is positively graded, $M_k = 0$ for $k \ll 0$. Hence the *Hilbert Series*

$$H_M(t) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{F}} M_k)\, t^k$$

is a well–defined element of $\mathbb{Z}[[t]][t^{-1}]$. Obviously it has no negative coefficients; for brevity we will call such a series *non–negative*.

By a classical result of Hilbert, this series may be written as a rational function of the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d}$$

where $Q_M \in \mathbb{Z}[t, t^{-1}]$ is a Laurent polynomial with $Q_M(1) \neq 0$, and d equals the Krull dimension of M.

Contrary to the dimension, the depth of a graded module cannot be read off its Hilbert series in general; there may be modules with the same Hilbert series, but different depths. So it makes sense to define

$$\operatorname{Hdep}(M) := \max \left\{ r \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists a f. g. gr. } R\text{–module } N \\ \text{with } H_N = H_M \text{ and } \operatorname{depth}(N) = r. \end{array} \right\},$$

this number will be called *Hilbert depth* of M. In the sequel we will show that $\operatorname{Hdep}(M)$ is given by an arithmetic invariant of H_M, and we will investigate the Hilbert depth of syzygies of the graded maximal ideal of R.

2. MAIN RESULT

We define the *positivity* of M by

$$\mathrm{p}(M) := \max \{ r \in \mathbb{N} \mid (1-t)^r H_M(t) \text{ is non–negative} \}.$$

This number turns out to coincide with the Hilbert depth of M:

Theorem 2.1. *Let M be a finitely generated graded module over the standard graded polynomial ring $\mathbb{F}[X_1, \dots, X_n]$, then* $\operatorname{Hdep}(M) = \mathrm{p}(M)$.

The inequality $\mathrm{Hdep}(M) \le \mathrm{p}(M)$ follows easily from standard facts about regular sequences. To prove the converse it is obviously enough to show that H_M always admits a decomposition

$$H_M(t) = \sum_{j=\mathrm{p}(M)}^{d} \frac{Q_j(t)}{(1-t)^j}$$

with non–negative numerators $Q_j \in \mathbb{Z}[t,t^{-1}]$, and this follows from

Lemma 2.2. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function of polynomial type with $f(k) = 0$ for $k \ll 0$. Then the formal Laurent series $H_f(t) = \sum_{k\in\mathbb{Z}} f(k)t^k$ admits a decomposition*

$$H_f(t) = \sum_{j=0}^{d} \frac{Q_j(t)}{(1-t)^j} \ \text{ with non–negative } Q_j \in \mathbb{Z}[t,t^{-1}].$$

Proof: See [2], Lemma 2.2. □

3. Syzygies of the *Maximal Ideal

The Hilbert depth of the ideal $\mathfrak{m} := (X_1,\ldots,X_n) \subset R$ is easily computed:

Theorem 3.1.

$$\mathrm{Hdep}(\mathfrak{m}) = \left\lfloor \frac{n+1}{2} \right\rfloor$$

Proof: For all $r \in \mathbb{N}$ we have

$$(1-t)^r H_{\mathfrak{m}}(t) = nt + \sum_{k=2}^{\infty} \left[\binom{n+k-1-r}{k} + (-1)^{k-1}\binom{r}{k} \right] t^k.$$

This series is non–negative for $r \le \lfloor \frac{n+1}{2} \rfloor$, and hence $\mathrm{p}(\mathfrak{m}) \ge \lfloor \frac{n+1}{2} \rfloor$. An inspection of the second coefficient also yields the reverse inequality. □

We also want to investigate the higher syzygy modules of $\mathbb{F} \cong R/\mathfrak{m}$. The Hilbert series of the j^{th} syzygy module $M(n,j)$ of $\mathbb{F}$ can be read off the Koszul complex; its numerator polynomial is

$$Q_{M(n,j)}(t) = \binom{n}{j} t^j - \binom{n}{j+1} t^{j+1} + \ldots + (-1)^{n-j} t^n.$$

The upper half of the resolution poses no problems: Since $\binom{n}{j} \ge \binom{n}{j+1}$ for $j \ge \lfloor \frac{n}{2} \rfloor$, one multiplication by $\frac{1}{1-t}$ is enough to remove all negative coefficients. Hence we have

Theorem 3.2. *Suppose $n > j \ge \lfloor n/2 \rfloor$. Then*

$$\mathrm{Hdep}\big(M(n,j)\big) = n-1.$$

In the lower half of the Koszul complex the situation is much more complicated. We have to determine the smallest $s \in \mathbb{N}$, such that

$$\frac{Q_{M(n,j)}(t)}{(1-t)^s} = \sum_{k=0}^{\infty} \left((-1)^k \binom{n-s}{j+k} + \sum_{t=1}^{s} \binom{n-t}{j-1} \binom{s-t+k}{s-t} \right) t^k$$

has non–negative coefficients. Since there is *no* summation formula available for the inner sum (a so–called ${}_3F_2$*–series*), this can probably not be done in closed form.

The quotient of the second, negative, term in the numerator polynomial by the first term yields an upper bound for the Hilbert depth of the low syzygies:

Lemma 3.3. *Let* $j < \lfloor n/2 \rfloor$. *Then*

$$\operatorname{Hdep}\big(M(n,j)\big) \leq n - \left\lceil \frac{n-j}{j+1} \right\rceil .$$

One might hope that this gives the correct value as it does in the cases $j = 1$ and $j \geq \lfloor n/2 \rfloor$. This is indeed true for $n \leq 22$, but for large n the upper bound of the lemma above is far from the truth, as the following asymptotic result shows:

Theorem 3.4. *For a fixed positive integer j, we have*

$$\operatorname{Hdep}\big(M(n,j)\big) = \frac{1}{2}n + \frac{1}{2}\sqrt{(j-1)n\log n} + \frac{1}{4}\sqrt{\frac{(j-1)n}{\log n}}\log\log n + o\left(\sqrt{\frac{n}{\log n}}\log\log n\right),$$

as $n \to \infty$.

Proof: See [1], Theorem 4.1. □

4. Generalization to Arbitrary Positive Gradings?

One may ask whether Theorem 2.1 can be extended to arbitrary positive $\mathbb{Z}$–gradings of R. The case where all indeterminates share a common degree $e > 1$ is easy:

Theorem 4.1. *Let* $R = \mathbb{F}[X_1,\ldots,X_n]$ *be graded with* $\deg(X_1) = \ldots = \deg(X_n) = e \geq 1$, *and let* $M \neq 0$ *be a finitely generated graded R–module. Then*

$$\operatorname{Hdep}(M) = \mathrm{p}_e(M) = \max\left\{r \in \mathbb{N} \mid (1-t^e)^r H_M(t) \text{ is positive}\right\}.$$

If the indeterminates are assigned different degrees, the situation gets much more complicated, since it is not clear what "positivity" should mean in this case. One idea could be to set $e := \operatorname{lcm}(e_1,\ldots,e_n)$ and to define $\mathrm{p}_e(M)$ as above. By the previous result, the assertion holds for modules over the subalgebra $S := \mathbb{F}[X_1^{e/e_1},\ldots,X_n^{e/e_n}]$, so one may try to give some *change–of–ring* argument. But it is doubtable whether this will work. Perhaps a completely new idea is required.

References

[1] W. Bruns, Chr. Krattenthaler, and J. Uliczka, *Stanley decompositions and Hilbert depth in the Koszul complex*, preprint, available on the Arxiv (AC/0909.0686).

[2] J. Uliczka, *Two Results on Hilbert Series of Graded Modules over Polynomial Rings*, preprint, submitted to manuscripta mathematica.

Buchsbaum* simplicial complexes

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1. INTRODUCTION

On of the central problems posed in geometric combinatorics is the classification of f-vectors of classes of simplicial complexes. Some classes are defined by combinatorial properties others by geometric or algebraic properties. Before we get to the specifics of certain classes, we provide the basic definitions.

An (abstract) simplicial complex Δ over groundset Ω is a subset $\Delta \subseteq 2^{\Omega}$ of the power set of Ω such that $B \subseteq A \in \Delta$ implies $B \in \Delta$. An element $A \in \Delta$ is a called a face of A. The dimension $\dim A$ of a face A is the cardinality of A minus 1. The dimension $\dim \Delta$ of Δ is the maximal dimension of one of its faces. The f-vector $f(\Delta)$ of Δ is the integer vector $f(\Delta) = (f_{-1}(\Delta), \ldots, f_{d-1}(\Delta))$, where $\dim \Delta = d-1$. Clearly, $f_{-1}(\Delta) = 1$ for all $\Delta \neq \emptyset$.

For any class $\mathscr{C}$ of simplicial complexes on can ask for a classification of the integer vectors $(f_{-1}, \ldots, f_{d-1}) \in \mathbb{N}^{d+1}$ in $f(\mathscr{C}) = \{f(\Delta) \mid \Delta \in \mathscr{C}\}$.

This question has been answered for various classes of simplicial complexes. For example if $\mathscr{C}$ is the class of all simplicial complexes then the Kruskal-Katona theorem (see [2]) provided the complete classification. For other classes of simplicial complexes it turns out that it is not suitable to use the f-vector itself for the classification but rather consider a linear transform of the f-factor. For this let $f(\Delta)(t) = \sum_{i=0}^{d} f_{i-1}(\Delta) t^{d-i}$ be the f-polynomial of Δ. Then the h-polynomial $h(\Delta)(t) = \sum_{i=0}^{d} h_i(\Delta) t^{d-i}$ is defined as $h(\Delta)(t) := f(\Delta)(t-1)$. The vector $h(\Delta) := (h_0(\Delta), \ldots, h_d(\Delta)$ is then called the h-vector of Δ. Clearly, by $f(\Delta)(t) = h(\Delta)(t+1)$ the h-vector and the f-vector carry the same information. Hence classifying the h-vectors $h(\mathscr{C}) := \{h(\Delta) \mid \Delta \in \mathscr{C}\}$ is equivalent to classifying the f-vectors $f(\mathscr{C})$. Nevertheless, the actual formulation of the classification may differ substantially in terms of technical complexity between h- and f-vectors.

The historically first class of simplicial complexes for which the h-vector has turned out to be the most suitable invariant for classification was the class of Cohen-Macaulay simplicial complexes. A simplicial complex Δ is called Cohen-Macaulay over a field k if its Stanley-Reisner ring $k[\Delta]$ is a Cohen-Macaulay ring (see [5] for a definition and background). Recall that the Stanley-Reisner ring $k[\Delta]$ of a simplicial complex over ground set Ω is the quotient of the polynomial ring $k[x_\omega \mid \omega \in \Omega]$ by the ideal generated by the monomials $\prod_{\omega \in N} x_\omega$ for $N \notin \Delta$. Now if $k[\Delta]$ is Cohen-Macaulay then by the definition of Cohen-Macaulay-ness and basic facts from commutative algebra it follows that an integer vector $(h_0, \ldots, h_d)$ is the h-vector of a Cohen-Macaulay simplicial complex if and only if there is standard graded 0-dimensional k-algebra $A = A_0 \oplus \cdots \oplus A_d$ such that $\dim_k A_i = h_i$. Hence the problem is reduced to the classification of Hilbert-series of 0-dimensional algebras which in turn is a result by Macaulay (see [5]).

By a result of Munkres [7] it is known that $k[\Delta]$ is Cohen-Macaulay over k if and only if $k[\Gamma]$ is Cohen-Macaulay for any triangulation Γ of the geometric realization $|\Delta|$ of Δ. Hence Cohen-Macaulay-ness is a topological property. Thus one can speak of a topological space X to be Cohen-Macaulay over a field k. One of the basic topological spaces that is Cohen-Macaulay over any field k is a sphere. Indeed triangulations of spheres enjoy the property of being Gorenstein* simplicial complexes over any field k. By this we again mean that any triangulation Δ of a sphere is Gorenstein* over k. A simplicial complex Δ is called Gorenstein* over k if and only if for all $A \in \Delta$ the simplicial homology of the link $\mathrm{lk}_\Delta(A) = \{B \mid A \cup B \in \Delta, A \cap B = \emptyset\}$ is 0 in homological dimensions $\neq \dim \mathrm{lk}_\Delta(A)$ and isomorphic to k in dimension $\dim \mathrm{lk}_\Delta(A)$. Again one can check [4] that being Gorenstein* is a topological property. The classification of f-vectors of simplicial complexes that are Gorenstein* over a field k is wide open. But there is a specific conjecture – the so called g-conjecture. It says that a sequence of numbers $(h_0, \ldots, h_d)$ is the h-vector of a Gorenstein* simplicial complex if and only if $h = 0 = 1$, $h_i = h_{d-i}$, $0 \leq i \leq d$ and the vector $(g_0, \ldots, g_{\lfloor \frac{d}{2} \rfloor})$ with $g_0 = h_0$ and $g_i = h_i - h_{i-1}$, $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$ is the h-vector of a Cohen-Macaulay simplicial complex. This conjecture is mainly motivated by the fact that the same classification holds for boundary complex of simplicial polytopes by a groundbreaking result of Stanley [10] and Billera & Lee [3].

If one wants to consider closed manifolds different from spheres then one has to leave the realm of Cohen-Macaulay simplicial complexes. Buchsbaum simplicial complexes can be defined through the ring theoretic notion of Buchsbaum-ness or the condition that all links inside a Buchsbaum simplicial complex are Cohen-Macaulay over k except for possibly the link of the empty simplex – which is the full complex.

Despite deep recent results by Novik & Swartz [8], [9] on the f-vector theory of Buchsbaum simplicial complexes there does not even seem to be a good conjecture of how f-vectors of Buchsbaum simplicial complexes look like. In addition there are many simplicial complexes inside the class of Buchsbaum simplicial complexes whose behavior is topologically much different from that of s manifold. This was the motivation for defining the class of Buchsbaum* simplicial complexes in [1].

2. Buchsbaum* simplicial complexes

Definition 2.1. Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex over a field k. The complex Δ is called Buchsbaum* over k if

$$\dim_k \widetilde{H}_{d-2}(|\Delta| - p; k) = \dim_k \widetilde{H}_{d-2}(|\Delta|; k) \tag{2.1}$$

holds for every $p \in |\Delta|$.

The results from [1] show that the notion of a Buchsbaum* complex provides a well behaved subclass of Buchsbaum complexes which contains all orientable manifolds. It behaves nicely in terms of of various enumerative, homological and graph theoretic properties. We summarize some of these results as follows.

The class of Buchsbaum* complexes is shown to be included in the class of doubly Buchsbaum complexes [1, Corollary 2.9] with nonvanishing top-dimensional homology [1, Corollary 2.4], to include all triangulations of orientable homology manifolds [1, Proposition 2.7] and to reduce to the class of doubly Cohen-Macaulay complexes, when restricted to the class of all Cohen-Macaulay complexes [1, Proposition 2.5]. The h'-vector is a natural analogue for Buchsbaum complexes of the h-vector of a Cohen-Macaulay complex (see e.g. [9]). A lower bound proved for h'-vectors of given dimension is given for Buchsbaum* flag complexes [1, Corollary 3.3]. Partially extending results of Kalai [6] on homology manifolds, the graph of a connected Buchsbaum* complex of dimension two or higher is shown to be generically d-rigid [1, Theorem 4.1]. This implies that the f-vector of a Buchsbaum* complex satisfies the inequalities of Barnette's lower bound theorem and that the first three entries of the g'-vector of a Buchsbaum* complex of dimension three or higher satisfy the conditions predicted by the g-conjecture.

The following result shows that Buchsbaum* complexes are still a very large class and go far beyond orientable homology manifolds.

Theorem 2.2 (Theorem 5.1 [1]). *Suppose that Δ is a $(d-1)$-dimensional simplicial complex and that there exist subcomplexes $\Delta_1, \Delta_2, \ldots, \Delta_m$ such that:*

(i) $\Delta = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$.

(ii) Δ_1 *is a $(d-1)$-dimensional orientable homology manifold over k.*

(iii) *For $2 \leq i \leq m$, Δ_i is a $(d-1)$-dimensional orientable homology manifold over k with boundary $\partial\Delta_i$ which has the following properties:*

  (a) $\partial\Delta_i$ *is a $(d-2)$-dimensional orientable homology manifold over k.*

  (b) $\partial\Delta_i = \Delta_i \cap (\Delta_1 \cup \cdots \cup \Delta_{i-1})$.

  (c) *The inclusion maps induce the zero homomorphisms*

$$\widetilde{H}_{d-2}(\partial\Delta_i; k) \to \widetilde{H}_{d-2}(\Delta_1 \cup \cdots \cup \Delta_{i-1}; k)$$

and

$$\widetilde{H}_{d-3}(\partial\Delta_i; k) \to \widetilde{H}_{d-3}(\Delta_1 \cup \cdots \cup \Delta_{i-1}; k).$$

Then Δ is Buchsbaum over k.*

REFERENCES

[1] C. Athanasiadis, V. Welker, *Buchsbaum* complexes*, preprint 2009, `arXiv:0910.3396`.

[2] L. Billera, A. Björner, *Face numbers of polytopes and complexes*, Goodman, Jacob E. (ed.) et al., Handbook of discrete and computational geometry, Boca Raton, FL: CRC Press. CRC Press Series on Discrete Mathematics and its Applications, pp. 291-310, 1997.

[3] L. Billera, C.W. Lee, *A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes*, J. Comb. Theory, Ser. A **31** (1981) 237-255.

[4] A. Björner, private communication, 2009.

[5] W. Bruns and J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[6] G. Kalai, *Rigidity and the lower bound theorem*, Invent. Math. **88** (1987), 125–151.

[7] J.R. Munkres, *Topological results in combinatorics*, Michigan Math. J. **31** (1984), 113–128.

[8] I. Novik and E. Swartz, *Gorenstein rings through face rings of manifolds*, Compos. Math. **144** (2009), 993-1000.

[9] I. Novik and E. Swartz, *Socles of Buchsbaum modules, complexes and posets*, Adv. Math. **222** (2009) 2059-2084.

[10] R.P. Stanley, *Combinatorial applications of the hard Lefschetz theorem*, Proc. Int. Congr. Math., Warszawa 1983, Vol. **1**, pp. 447-453, 1984.

On the $\mathbb{A}^1$-Fundamental Groups of Smooth Toric Varieties

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In the talk, I explained some results on the structure of the $\mathbb{A}^1$-fundamental groups of smooth toric varieties which are published in [Wen09]. In these notes, I will state the main results and illustrate them in the case of the projective spaces. Proofs of the results can be found in [Wen09]. I would like to note that similar results have been obtained by Asok and Doran [AD07].

The philosophy which will be visible throughout this introduction is that the structure of the $\mathbb{A}^1$-fundamental groups is basically the same as the structure of the fundamental groups of *real* toric varieties – one just has to replace $\mathbb{Z}/2\mathbb{Z}$ by the multiplicative group $\mathbb{G}_m$, and the category of groups by the category of strongly $\mathbb{A}^1$-invariant sheaves of groups. Before being able to state the result, I will introduce a bit of notation for toric varieties and explain what $\mathbb{A}^1$-homotopy theory is.

Toric Varieties: A good introduction to the theory of toric varieties is the book [Ful93]. I will not give complete definitions, but merely introduce notation for fans, toric varieties etc.

Let $N \cong \mathbb{Z}^n$ be a lattice, and Δ be a fan in $N \otimes_{\mathbb{Z}} \mathbb{R}$. Cones of the fan will be denoted by σ, and the set of k-dimensional cones of Δ is denoted by $\Delta(k)$. A fan Δ is called *regular* if each cone $\sigma \in \Delta$ is generated by vectors $v_1, \dots, v_k$ which can be completed to a $\mathbb{Z}$-basis of N. Over each base scheme S, one can associate to a regular fan Δ an S-scheme $X(\Delta)$, cf. [Cox95a]. This is what the term *smooth toric variety* means here, although that is quite an abuse of language.

Example 1. The projective spaces $\mathbb{P}^n$ are given by the following fans. The lattice N is the standard lattice $\mathbb{Z}^n$ generated by $e_1, \dots, e_n$. We set $e_0 = -\sum_{i=1}^n e_i$. The fan Δ is given by the cones generated by n-element subsets of $\{e_0, \dots, e_n\}$.

In [Cox95b], the homogeneous coordinate ring of projective space was generalized to toric varieties. This yields a quasi-affine variety which is a torus-covering of the original toric variety, generalizing the quotient presentation $\mathbb{A}^{n+1} \to \mathbb{P}^n$. In fact, this covering can be described as the toric variety associated to the fan Δ, but living in the lattice $\mathbb{Z}^{\Delta(1)}$ which is generated by elements corresponding to one-dimensional cones. We will denote this fan by $\widetilde{\Delta}$. It was shown in [Cox95b] that in case Δ is regular, the toric variety $X(\Delta)$ is the geometric quotient of $X(\widetilde{\Delta})$ by a suitable torus action.

Fundamental Groups of Real Projective Space: As an example we explain the structure of the fundamental group of real projective space. This will serve as a blueprint for the description of the $\mathbb{A}^1$-fundamental groups. The general description of fundamental groups of real toric varieties can be found in [Uma04].

Example 2. We consider the real projective line $\mathbb{RP}^1$. This is homeomorphic to S^1, and in particular we have

$$\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}.$$

The homogeneous coordinate ring has two variables, and there is a fibre bundle $\mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^1$ with fibre $\mathbb{R} \setminus \{0\}$. This induces a short exact sequence of fundamental groups

$$0 \to \pi_1(\mathbb{R}^2 \setminus \{0\}) \to \pi_1(\mathbb{RP}^1) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

There is another fibre sequence

$$\mathbb{R}^2 \setminus \{0\} \to \mathbb{RP}^\infty \vee \mathbb{RP}^\infty \to \mathbb{RP}^\infty \times \mathbb{RP}^\infty,$$

which induces an exact sequence

$$0 \to \pi_1(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 0.$$

One can show that the kernel of the second map is the commutator subgroup of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, which is isomorphic to $\mathbb{Z}$. This is an overly complicated proof that $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$, but it is this description which generalizes to the motivic setting.

Example 3. We now consider real projective space $\mathbb{RP}^n$ for $n \geq 2$. Again the covering $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ induces an exact sequence

$$0 \to \pi_1(\mathbb{R}^{n+1} \setminus \{0\}) \to \pi_1(\mathbb{RP}^n) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

This time, however, $\mathbb{R}^{n+1} \setminus \{0\}$ is simply-connected, which implies

$$\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}.$$

Main Results: $\mathbb{A}^1$-homotopy theory is a homotopy theory for algebraic varieties developed by Morel and Voevodsky, cf. [MV99]. Putting all the complicated homotopical algebra aside, $\mathbb{A}^1$-homotopy theory is the study of the homotopy category of

a Bousfield localization of a model structure of simplicial sheaves on the category of smooth schemes. Unwinding this definition yields a notion of $\mathbb{A}^1$-fundamental groups, which are sheaves of groups on the category of smooth schemes which satisfy an additional condition called strong $\mathbb{A}^1$-invariance, cf. [Mor06].

The results below provide some structural information about the $\mathbb{A}^1$-fundamental groups of smooth toric varieties. The basic techniques used to prove these results are statements about the behaviour of homotopy limits and colimits in $\mathbb{A}^1$-homotopy theory [Wen07] and an $\mathbb{A}^1$-homotopy version of the van-Kampen theorem [Mor06]. Once a working theory of homotopy limits and colimits is available, the proof for the real toric varieties can be adapted to $\mathbb{A}^1$-homotopy theory. Note that the results below hold over any regular base scheme S.

Proposition 4. Let N be a lattice, and let Δ be a regular fan. If $\operatorname{span}\Delta(1) = N \otimes_{\mathbb{Z}} \mathbb{R}$, then $X(\Delta)$ is $\mathbb{A}^1$-connected.

We now consider only $\mathbb{A}^1$-connected $X(\Delta)$ with any choice of base point.

Proposition 5. The torus covering of [Cox95b] induces an $\mathbb{A}^1$-local fibre sequence. This implies that there is a short exact sequence of $\mathbb{A}^1$-fundamental groups

$$0 \to \pi_1^{\mathbb{A}^1}(X(\widetilde{\Delta})) \to \pi_1^{\mathbb{A}^1}(X(\Delta)) \to \mathbb{G}_m^d \to 0,$$

where $d = \#\Delta(1) - \operatorname{rk} N$.

A motivic analogue of Davis-Januszkiewicz spaces can be developed, cf. [Wen09]. The $\mathbb{A}^1$-homotopy version of the van-Kampen theorem then implies the following result. For a precise formulation see [Wen09].

Theorem 6. The $\mathbb{A}^1$-fundamental group sheaf of $X(\widetilde{\Delta})$ is the commutator subgroup sheaf of a graph product of copies of $\mathbb{G}_m$. The graph is given by the one- and two-dimensional cones of Δ: the vertices are the one-dimensional cones, and two vertices are connected by an edge if the corresponding one-dimensional cones span a two-dimensional cone. The graph product is taken in the category of strongly $\mathbb{A}^1$-invariant sheaves of groups on the category of smooth schemes.

Corollary 7. Let S be a regular base scheme, and $X(\Delta)$ the toric variety over S associated to a regular fan Δ.

- The $\mathbb{A}^1$-fundamental group of $X(\Delta)$ depends only on the one- and two-dimensional cones of the fan Δ.
- If any two one-dimensional cones of Δ span a two-dimensional cone, then the $\mathbb{A}^1$-fundamental group of $X(\Delta)$ is a torus.

Example 8. We consider the projective line $\mathbb{P}^1$. The $\mathbb{G}_m$-torsor $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ induces a short exact sequence of sheaves of groups

$$0 \to \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) \to \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m \to 0$$

There is a fibre sequence

$$0 \to \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^\infty \vee \mathbb{P}^\infty \to \mathbb{P}^\infty \times \mathbb{P}^\infty \to 0,$$

which shows that the $\mathbb{A}^1$-fundamental group of $\mathbb{A}^2 \setminus \{0\}$ is the $\mathbb{A}^1$-localization of the commutator subgroup sheaf of the free product $\mathbb{G}_m * \mathbb{G}_m$. It is noteworthy that this object is abelian, in fact, Morel has shown in [Mor06] that it is isomorphic to the second Milnor-Witt K-theory K_2^{MW}.

Example 9. We now consider $\mathbb{P}^n$ for $n \geq 2$. There is a $\mathbb{G}_m$-torsor $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The above results imply that $\mathbb{A}^{n+1} \setminus \{0\}$ is $\mathbb{A}^1$-simply connected, and therefore

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) \cong \mathbb{G}_m.$$

References

[AD07] A. Asok and B. Doran. $\mathbb{A}^1$-homotopy groups, excision and solvable quotients. Adv. Math. 221 (2009), 1144–1190.

[Cox95a] D.A. Cox. The functor of a smooth toric variety. Tohoku Math. J. 47 (1995), 251–262.

[Cox95b] D.A. Cox. The homogeneous coordinate ring of a toric variety. J. Algebraic Geom. 4 (1995), 17–50.

[Ful93] W.E. Fulton. Introduction to toric varieties. Annals of Mathematics Studies 131. Princeton University Press (1993).

[Mor06] F. Morel. $\mathbb{A}^1$-algebraic topology over a field. 2006. Preprint.

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. Publ. Math. Inst. Hautes Études Sci. 90 (1999), 45–143.

[Uma04] V. Uma. On the fundamental group of real toric varieties. Proc. Indian Acad. Sci. Math. Sci. 14 (2004), 15–31.

[Wen07] M. Wendt. On fibre sequences in motivic homotopy theory. PhD thesis, Universität Leipzig (2007).

[Wen09] M. Wendt. On the $\mathbb{A}^1$-fundamental groups of smooth toric varieties. Adv. Math. 2009 (doi:10.1016/j.aim.2009.08.004)