On h-vectors of broken circuit complexes

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Outline

Chromatic polynomials

$G = (V, E)$: a graph, $|V| = n$.

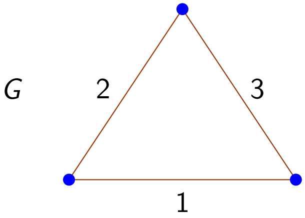
- Birkhoff [Bir12]: For $t \in \mathbb{N}$, let $\chi(G, t)$ be the number of proper colorings of G with t colors, i.e., the number of maps $\gamma : V \to \{1, 2, \ldots, t\}$ such that $\gamma(u) \neq \gamma(v)$ whenever $\{u, v\} \in E$. Then $\chi(G, t)$ is a polynomial in t of degree n, called the chromatic polynomial of G:

$$\chi(G, t) = f_0 t^n - f_1 t^{n-1} + f_2 t^{n-2} - \cdots + (-1)^{n-1} f_{n-1} t.$$

- Whitney [Wh32a]: Assign a linear order to E. A broken circuit is a cycle of G with the least edge removed. Then

$$f_i = \sharp\{i\text{-subsets of } E \text{ which contain no broken circuit}\}.$$

Example



- Broken circuit: $\{2,3\}$.
- $f_0 = \sharp\{\emptyset\} = 1$, $f_1 = \sharp\{\{1\},\{2\},\{3\}\} = 3$, $f_2 = \sharp\{\{1,2\},\{1,3\}\} = 2$.
- $\chi(G,t) = t^3 - 3t^2 + 2t = t(t-1)(t-2)$.

Broken circuit idea

- Rota [Rot64]: extended Whitney's formula to characteristic polynomials of matroids.
- Wilf [Wil76]: the collection of all subsets of E which contain no broken circuit forms a simplicial complex.
- Brylawski [Bry77]: defined broken circuit complexes of matroids.

Matroids

Whitney [Wh35]: A matroid **M** consists of a finite ground set E and a non-empty collection $\mathcal{I}$ of subsets of E, called independent sets, satisfying the following conditions:

1. subsets of independent sets are independent,
2. for every subset X of E, all maximal independent subsets of X have the same cardinality, called the rank of X.

A subset of E is called dependent if it is not a member of $\mathcal{I}$. Minimal dependent sets are called circuits. The rank of E is also called the rank of **M** and denoted by $r(\mathbf{M})$.

Examples

1. Linear/representable matroids: Let W be a vector space over a field K and E a finite subset of W. The linear matroid of E:
   - ground set: E,
   - independent sets: linearly independent subsets of E.

   Matroids of this type are called representable over K.
2. Cycle/graphic matroids: Let G be a graph with edge set E. The cycle matroid $\mathbf{M}(G)$:
   - ground set: E,
   - independent sets: subsets of E containing no cycle.

   Matroids of this type are called graphic matroids.

Broken circuit complexes

Let $\mathbf{M}$ be a matroid on the ground set E. Assign a linear order $<$ to E. A broken circuit of $\mathbf{M}$ is a subset of E of the form $C - e$, where C is a circuit of $\mathbf{M}$ and e is the least element of C. The broken circuit complex of $(\mathbf{M}, <)$, denoted $BC_{<}(M)$ (or briefly $BC(\mathbf{M})$), is defined by

$$BC(\mathbf{M}) = \{F \subseteq E \mid F \text{ contains no broken circuit}\}.$$

Broken circuit complexes

- $\dim BC(\mathbf{M}) = r(\mathbf{M}) - 1$.
- $BC(\mathbf{M})$ is a cone with apex e_0, where e_0 is the smallest element of E. The restriction of $BC(\mathbf{M})$ to $E - e_0$ is called the reduced broken circuit complex, denoted $\overline{BC}(\mathbf{M})$.
- Provan [Pro77]: $BC(\mathbf{M})$ is shellable.

Combinatorial aspect of broken circuit complexes

- Let $r = r(\mathbf{M})$. Let $\chi(\mathbf{M}, t) = \sum_{X \subseteq E} (-1)^{|X|} t^{r - r(X)}$ be the characteristic polynomial of $\mathbf{M}$. Then Rota [Rot64]: $\chi(\mathbf{M}, t) = f_0 t^r - f_1 t^{r-1} + \cdots + (-1)^r f_r$, where $(f_0, f_1, \ldots, f_r)$ is the f-vector of $BC(\mathbf{M})$: $f_i = \sharp$ faces of $BC(\mathbf{M})$ of cardinality i.
- $\chi(G, t) = t^{c(G)} \chi(\mathbf{M}(G), t)$, where $c(G)$ is the number of connected components of G.
- The h-vector $(h_0, h_1, \ldots, h_r)$ of $BC(\mathbf{M})$: $\sum_{i=0}^{r} f_i (t-1)^{r-i} = \sum_{i=0}^{r} h_i t^{r-i}$, or equivalently,

$$f_i = \sum_{j=0}^{i} \binom{r-j}{i-j} h_j, \quad i = 0, \ldots, r,$$

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{r-j}{i-j} f_j, \quad i = 0, \ldots, r.$$

Combinatorial aspect of broken circuit complexes

- Wilf [Wil76]: Which polynomials are chromatic?
- Problem: Characterize f-vectors (h-vectors) of broken circuit complexes.
- Conjecture (Welsh [Wel76]): Let $(f_0, f_1, \ldots, f_r)$ be the f-vector of $BC(\mathbf{M})$. Then $f_0, f_1, \ldots, f_r$ form a log-concave sequence, i.e., $f_{i-1}f_{i+1} \leq f_i^2$ for $0 < i < r$.
  $\rightsquigarrow$ solved by Adiprasito-Huh-Katz.
- Conjecture (Hoggar [Hog74]): The h-vector of $BC(\mathbf{M})$ is a log-concave sequence.
  $\rightsquigarrow$ verified by Huh [Huh15] for the case $\mathbf{M}$ is representable over a field of characteristic zero.

Algebraic aspect of broken circuit complexes

The broken circuit complex of the underlying matroid of a hyperplane arrangement induces

- a basis for the Orlik-Solomon algebra (Orlik-Solomon [OS80], Björner [Bjo82], Gel'fand-Zelevinsky [GZ86], Jambu-Terao [JT89]).
- a basis for the Orlik-Terao algebra (Proudfoot-Speyer [PS06]).

Outline

Hyperplane arrangements

- A hyperplane arrangement in a K-vector space V is a finite set of linear hyperplanes

$$\mathcal{A} = \{H_1, \ldots, H_n\},$$

where $H_i = \ker \alpha_i$ with $\alpha_i \in V^*$.

- The linear matroid of $\alpha_1, \ldots, \alpha_n$ is called the underlying matroid of $\mathcal{A}$, denoted by $\mathbf{M}(\mathcal{A})$.
- Problem: Decide whether a certain property of $\mathcal{A}$ is combinatorial, i.e., determined by $\mathbf{M}(\mathcal{A})$.

Hyperplane arrangements

- Zaslavsky [Zas75]: Let $\mathcal{A}$ be a real arrangement. Then the number of regions of the complement $\mathcal{M}(\mathcal{A}) := V - \bigcup_{i=1}^{n} H_i$ is $|\chi(\mathbf{M}(\mathcal{A}), -1)|$.
- Orlik-Solomon [OS80]: If $\mathcal{A}$ is a complex arrangement, then the cohomology ring of $\mathcal{M}(\mathcal{A})$ is isomorphic to the so-called Orlik-Solomon algebra of $\mathcal{A}$, which is combinatorially determined.
- Rybnikov [Ryb11]: The fundamental group of $\mathcal{M}(\mathcal{A})$ is not combinatorial.
- Conjecture (Terao [Te80]): Freeness of arrangements is combinatorial.

2-formal arrangements

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $H_i = \ker \alpha_i$, $S = K[x_1, \ldots, x_n]$ a polynomial ring. The relation space $F(\mathcal{A})$ of $\mathcal{A}$ is the kernel of the K-linear map

$$S_1 = \bigoplus_{i=1}^{n} Kx_i \to V^*, \; x_i \mapsto \alpha_i \quad \text{for} \quad i = 1, \ldots, n.$$

Thus relations come from dependencies: if $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ is dependent and $\sum_{j=1}^{m} a_j \alpha_{i_j} = 0$, then $r = \sum_{j=1}^{m} a_j x_{i_j} \in F(\mathcal{A})$.

- Falk-Randell [FR86]: $\mathcal{A}$ is called 2-formal if $F(\mathcal{A})$ is spanned by relations of length 3 (i.e., having 3 nonzero coefficients).
- Yuzvinsky [Yuz93]: 2-formality is not combinatorial.
- Schenck-Tohaneanu [ST09]: characterized 2-formality in terms of the Orlik-Terao.

The Orlik-Terao algebra

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $H_i = \ker \alpha_i$. The Orlik-Terao algebra of $\mathcal{A}$ is the subalgebra of the field of rational functions on V generated by reciprocals of the α_i:

$$C(\mathcal{A}) := K[1/\alpha_1, \ldots, 1/\alpha_n].$$

Write $C(\mathcal{A}) = K[x_1, \ldots, x_n]/I(\mathcal{A})$, then $I(\mathcal{A})$ is the Orlik-Terao ideal of $\mathcal{A}$.

- Orlik-Terao [OT94]: answered a question of Aomoto in the context of hypergoemetric functions.
- Schenck-Tohaneanu [ST09]: characterized 2-formality in terms of the Orlik-Terao.
- Sanyal-Sturmfels-Vinzant [SSV13]: $C(\mathcal{A})$ is the coordinate ring of the reciprocal plane, which relates to a model in theoretical neuroscience.

The broken circuit complex and the Orlik-Terao algebra

- Proudfoot-Speyer [PS06]: Let $\mathcal{A}$ be an arrangement. Then the Stanley-Reisner ideal of any broken circuit complex of $\mathbf{M}(\mathcal{A})$ is an initial ideal of $I(\mathcal{A})$. In particular, $C(\mathcal{A})$ is a Cohen-Macaulay ring.
- Question: When are the broken circuit complex and the Orlik-Terao algebra complete intersections or Gorenstein?

Gorenstein and complete intersection properties

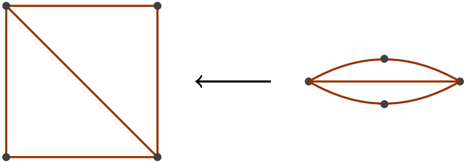
L. [Le14]:

- Let $\mathbf{M}$ be a matroid. Then $BC(\mathbf{M})$ is Gorenstein iff it is a complete intersection.
- Let $\mathcal{A}$ be an arrangement. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(\mathbf{M}(\mathcal{A}))$ with s being the largest index i such that $h_i \neq 0$. Then the following conditions are equivalent:
  1. $C(\mathcal{A})$ is Gorenstein.
  2. $h_i = h_{s-i}$ for $i = 0, \ldots, s$.
  3. $h_0 = h_s$ and $h_1 = h_{s-1}$.
  4. Every connected component of $\mathbf{M}(\mathcal{A})$ is either a coloop or a parallel connection of circuits.
  5. There exists an ordering $<$ such that $BC_<(\mathbf{M}(\mathcal{A}))$ is Gorenstein/a complete intersection.
  6. $C(\mathcal{A})$ is a complete intersection.

Outline

Series-parallel networks

- A 2-connected graph is a series-parallel network if it can be obtained from the complete graph K_2 by subdividing and duplicating edges.
- Example:



Series-parallel networks

- Dirac [Di52], Duffin [Duf65]: A loopless, 2-connected graph is a series-parallel network iff it has no subgraph that is a subdivision of K_4.
- Brylawski [Bry71]: Let G be a 2-connected graph. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(\mathbf{M}(G))$ with $h_s \neq 0$. Then G is a series-parallel network iff $h_s = 1$ (i.e., $h_s = h_0$).

Ear decompositions

- Let G be a loopless connected graph. An ear decomposition of G is a partition of the edges of G into a sequence of ears $\pi_1, \pi_2, \ldots, \pi_n$ such that:

(ED1) π_1 is a cycle and each π_i is a simple path (i.e., a path that does not intersect itself) for $i \geq 2$,

(ED2) each end vertex of π_i, $i \geq 2$, is contained in some π_j with $j < i$,

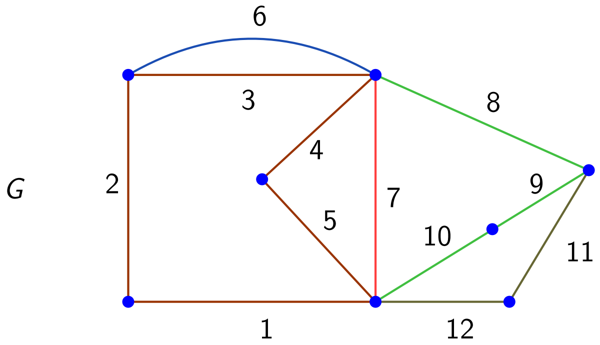
(ED3) no internal vertex of π_i is in π_j for any $j < i$.

- Whitney [Wh32b]: A graph with at least 2 edges admits an ear decomposition iff it is 2-connected.

Nested ear decompositions

- Let $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be an ear decomposition of a graph G. Then π_i is called nested in π_j, $j < i$, if both end vertices of π_i belong to π_j and at least one of them is an internal vertex of π_j.
- If π_i is nested in π_j, the nest interval of π_i in π_j is the path in π_j between the two end vertices of π_i.
- The ear decomposition Π is called nested if the following conditions hold:
  - (N1) for each $i > 1$ there exists $j < i$ such that π_i is nested in π_j,
  - (N2) if π_i and π_k are both nested in π_j, then either their nest intervals in π_j are disjoint, or one nest interval contains the other.

Example



A nested ear decomposition of G: $\pi_1 = \{1, 2, 3, 4, 5\}$, $\pi_2 = \{6\}$,$\pi_3 = \{7\}$,$\pi_4 = \{8, 9, 10\}$,$\pi_5 = \{11, 12\}$.

Nested ear decompositions

Eppstein [Epp92]: Let G be a 2-connected graph. Then the following conditions are equivalent:

1. G is a series-parallel network;
2. G has a nested ear decomposition;
3. every ear decomposition of G is nested.

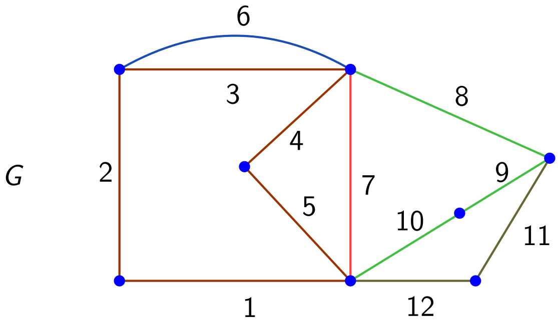
Nested ear decompositions and h-vectors of BCC

- Let $\Pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a nested ear decomposition of a series-parallel network G. If I is a nest interval, set

  $$\lambda(I) := \min\{\text{length}(I), \text{length}(\pi_i) \mid I \text{ is the nest interval of } \pi_i\}.$$

- Define $p(\Pi; G)$ = number of nest interval I such that $\lambda(I) > 1$.
- L. [Le16]: Let G be a series-parallel network. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(\mathbf{M}(G))$ with $h_s \neq 0$. Then $h_{s-1} - h_1 = p(\Pi; G)$ for any ear decomposition Π of G.

Example



- Nested intervals: $I_1 = \{3\},\ I_2 = \{4, 5\},\ I_3 = \{9, 10\}$.
- $\lambda(I_1) = \lambda(I_2) = 1,\ \lambda(I_3) = 2 \Rightarrow h_5 - h_1 = 1$.

Outline

Independence complexes

- Let $\mathbf{M}$ be a matroid with collection of independent sets $\mathcal{I}$. Then $\mathcal{I}$ forms a simplicial complex, called the independence complex of $\mathbf{M}$, denoted by $IN(\mathbf{M})$.
- $BC(\mathbf{M}) \subseteq IN(\mathbf{M})$.
- Brylawski [Bry77]: Given a matroid $\mathbf{M}$, there exists a matroid $\mathbf{M}'$ such that $IN(\mathbf{M}) = \overline{BC}(\mathbf{M}')$.
- $\{h\text{-vectors of independence complexes}\} \subset \{h\text{-vectors of broken circuit complexes}\}$.

h-vectors of independence complexes

- Problem: Characterize h-vectors (f-vectors) of independence complexes.
- Conjecture (Stanley [Sta77]): h-vectors of independence complexes are pure O-sequences.
- Conjecture (Hibi [Hi92]): Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $IN(\mathbf{M})$. Then

$$h_0 \leq h_1 \leq \cdots \leq h_{\lfloor s/2 \rfloor},$$
$$h_i \leq h_{s-i} \text{ for } i = 0, \ldots, \lfloor s/2 \rfloor.$$

- Chari [Cha97]: proved Hibi's conjecture.

h-vectors of broken complexes

- Conjecture (Swartz [Swa03]): Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(\mathbf{M})$ with $h_s \neq 0$. Then

$$h_i \leq h_{s-i} \text{ for } i = 0, \ldots, \lfloor s/2 \rfloor.$$

- L. [Le16]: Let $\mathbf{M} = \mathbf{M}(G)$, where G is a series-parallel network. Let $(h_0, h_1, \ldots, h_s)$ be the h-vector of $BC(\mathbf{M})$ with $h_s \neq 0$.
  1. If $h_{s-1} - h_1 = 1$, then $h_i \leq h_{s-i}$ for $i = 0, \ldots, \lfloor s/2 \rfloor$.
  2. $h_2 \leq h_{s-2}$ (when $s \geq 4$).

Thank you!

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