On the toric ideal of a matroid and related combinatorial problems

Michał Lasoń

Institute of Mathematics of the Polish Academy of Sciences

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- ... and by many other ways (circuits, flats, hyperplanes)

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By the symmetric exchange property from bases B, B' we get bases

$$D = (B \setminus e) \cup f \text{ and } D' = (B' \setminus f) \cup e$$

for some $e \in B$ and $f \in B'$.

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- *WEAK: I_M is generated by quadratic binomials*
- *CLASSIC: I_M is generated by quadratic binomials corresponding to symmetric exchanges*
- *STRONG: I_M in noncommutative ring S_M is generated by quadratic binomials corresponding to symmetric exchanges*

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  CLASSIC up to saturation for arbitrary matroids

cyclic ordering conjecture

Conjecture (Kajitani, Ueno, Miyano '88)

Let $M = (E, r)$ be a matroid. The following conditions are equivalent:

- *for each $\emptyset \neq A \subset E$ the inequality $\frac{|A|}{r(A)} \leqslant \frac{|E|}{r(E)}$ holds*
- *there exists a cyclic ordering – it is possible to place elements of E on a circle in such a way that any $r(E)$ cyclically consecutive elements form a basis*

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Theorem (van den Heuvel, Thomassé '12)

If $|E|$ and $r(E)$ are coprime, then cyclic ordering conjecture holds for $M = (E, r)$.

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Graph $\mathfrak{B}_2(M)$ with

- vertices – pairs of bases (B_1, B_2) which sum to E,
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Conjecture (cyclic ordering conjecture for a 2-matroid)

There exist complementary bases B_1, B_2 in M, such that vertices (B_1, B_2) and (B_2, B_1) in the graph $\mathfrak{B}_2(M)$ are connected by a path of length at most $r(E)$.

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Conjecture (Farber, Richter, Shank '85)

For every 2-matroid M the graph $\mathfrak{B}_2(M)$ is connected.

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Proposition (Blasiak '08)

- *WEAK $\iff$ for $k \geqslant 3$ graph $\mathfrak{B}_k$ is connected*

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- *STRONG* $\iff$ *for* $k \geqslant 2$ *graph* $\mathfrak{B}_k$ *is connected*

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Conjecture (cyclic for 2-matroid)

Some $(B_1, B_2), (B_2, B_1)$ *are connected by a path* $\leqslant r(E)$.

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Graph $\mathfrak{B}_2(M)$ has diameter $r(E)$.

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Thank you!