CAP - Categories, Algorithms, and Programming

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We call this concept **categorical programming**.

Outline

1 Motivation

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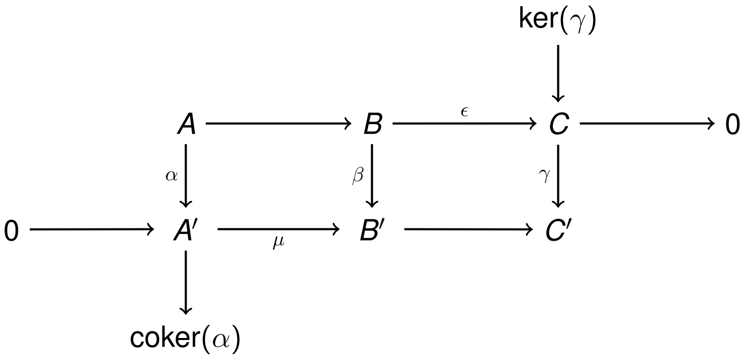
2 Flavor of categorical programming in CAP

Section 1

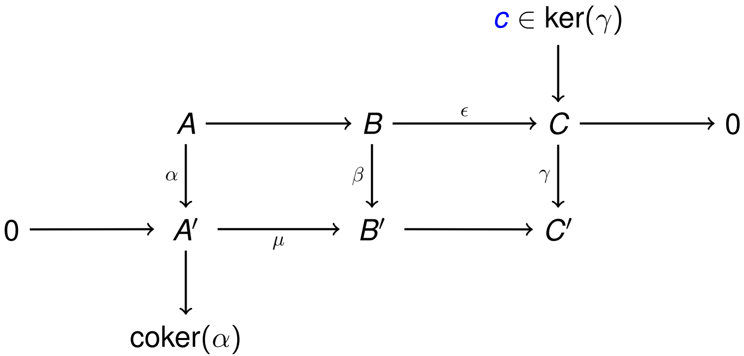
Motivation

Connecting Homomorphism in the Snake Lemma



Wanted: ker(γ) $\xrightarrow{\partial}$ coker(α).

Connecting Homomorphism in the Snake Lemma



Start: $c \in \ker(\gamma)$.

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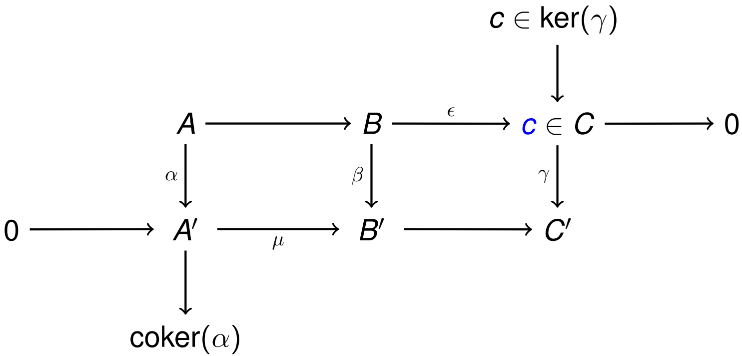
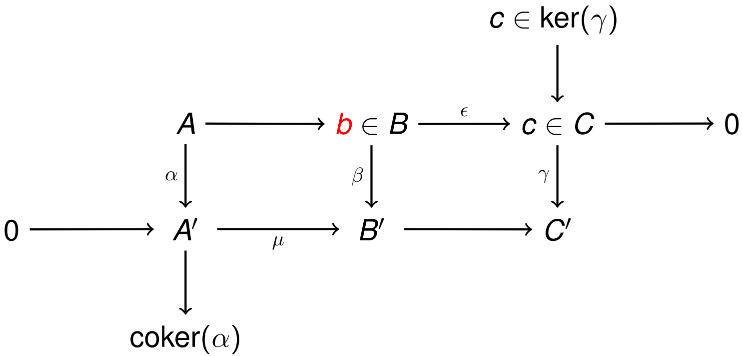


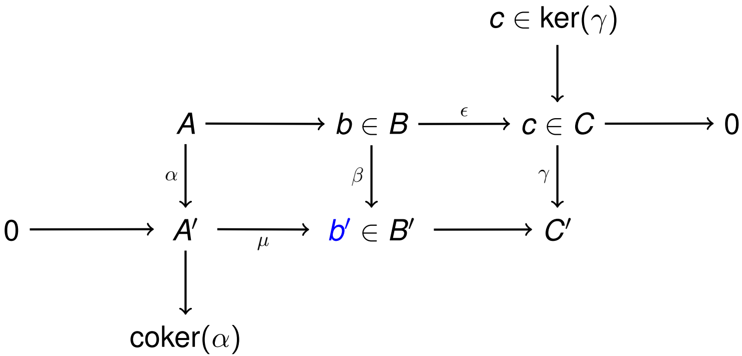
This lies in C.

Connecting Homomorphism in the Snake Lemma



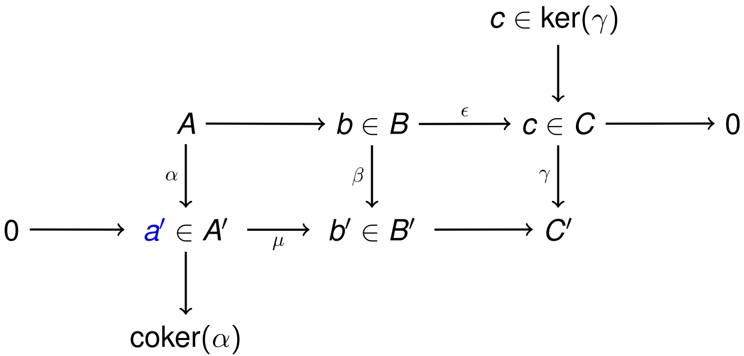
Choose: $b \in \epsilon^{-1}(\{c\})$.

Connecting Homomorphism in the Snake Lemma



Map: $b \overset{\beta}{\mapsto} b'$.

Connecting Homomorphism in the Snake Lemma



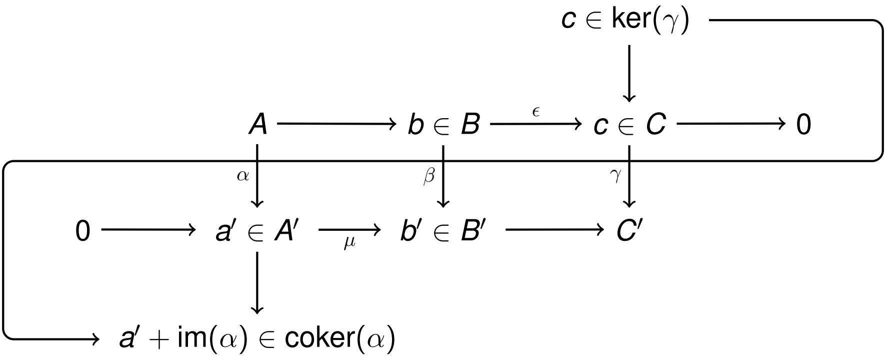
Compute: $a' \in \mu^{-1}(b')$.

Connecting Homomorphism in the Snake Lemma

$$
\begin{array}{ccccccccc}
 & & & & & & c \in \ker(\gamma) & & \\
 & & & & & & \downarrow & & \\
 & & A & \longrightarrow & b \in B & \xrightarrow{\epsilon} & c \in C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & a' \in A' & \xrightarrow[\mu]{} & b' \in B' & \longrightarrow & C' & & \\
 & & \downarrow & & & & & & \\
 & & {\color{blue} a' + \mathrm{im}(\alpha)} \in \mathrm{coker}(\alpha) & & & & & &
\end{array}
$$

Map: $a' \mapsto a' + \mathrm{im}(\alpha)$.

Connecting Homomorphism in the Snake Lemma



Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$.

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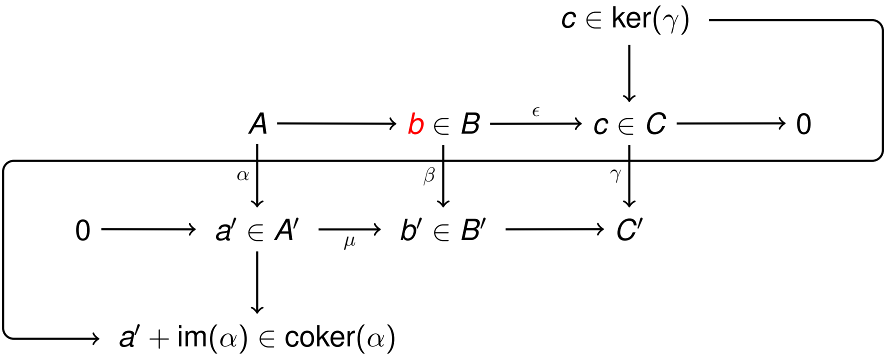
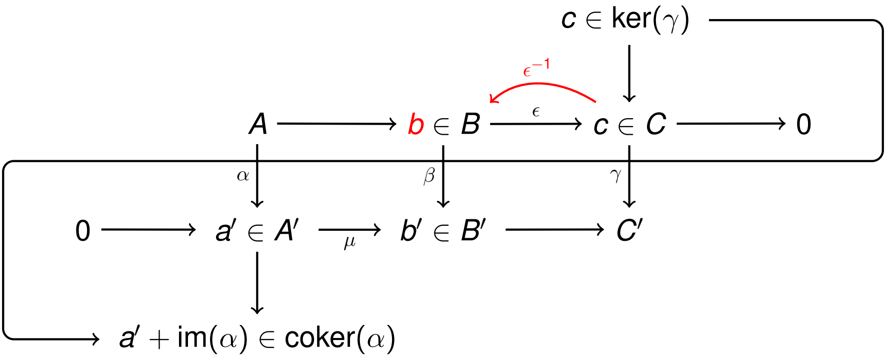


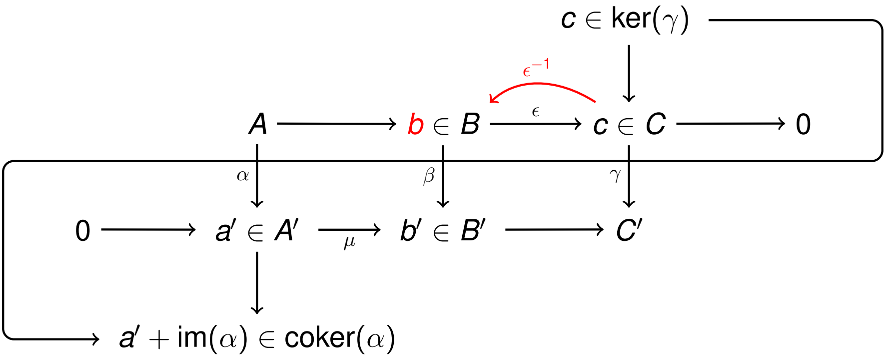
Result: $c \overset{\partial}{\mapsto} a' + \mathrm{im}(\alpha)$. Independent of the **choice**.

Connecting Homomorphism in the Snake Lemma



Any **right inverse** can be used.

Connecting Homomorphism in the Snake Lemma



Q: What if ϵ has no right inverse?

Generalized Morphism Category

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$\mathbf{A}$

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$\mathbf{A}$ abelian category

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$$\mathbf{A} \text{ abelian category} \xrightarrow{\text{CAP}}$$

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Properties of $G(\mathbf{A})$

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- $\mathbf{A} \subseteq G(\mathbf{A})$.

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Properties of $G(\mathbf{A})$

- $\mathbf{A} \subseteq G(\mathbf{A})$.
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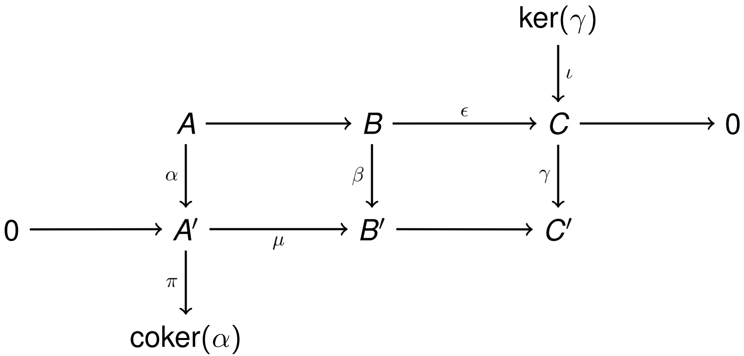
Properties of $G(\mathbf{A})$

- $\mathbf{A} \subseteq G(\mathbf{A})$.
- Every monomorphism has a **canonical** left inverse.
- Every epimorphism has a **canonical** right inverse.

Snake Lemma Revisited



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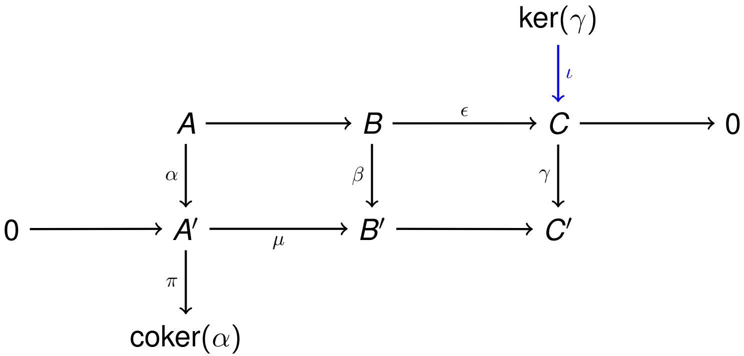

ι

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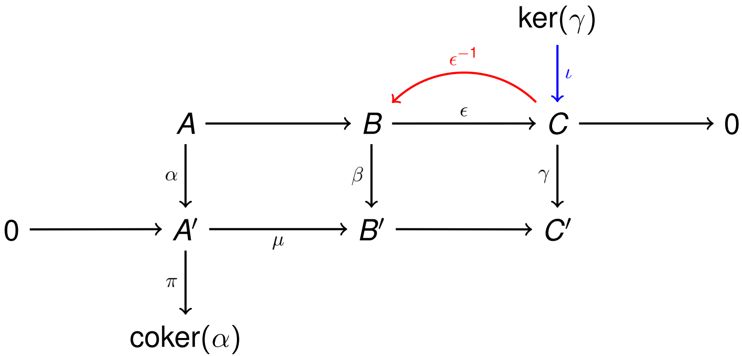

$$\epsilon^{-1} \circ \iota$$

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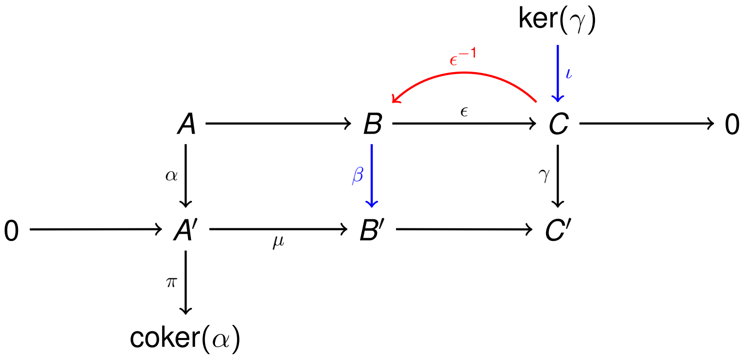

$$\beta \circ \epsilon^{-1} \circ \iota$$

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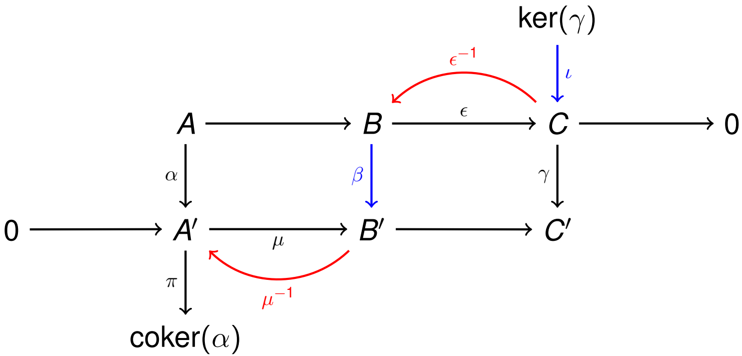

$$\mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

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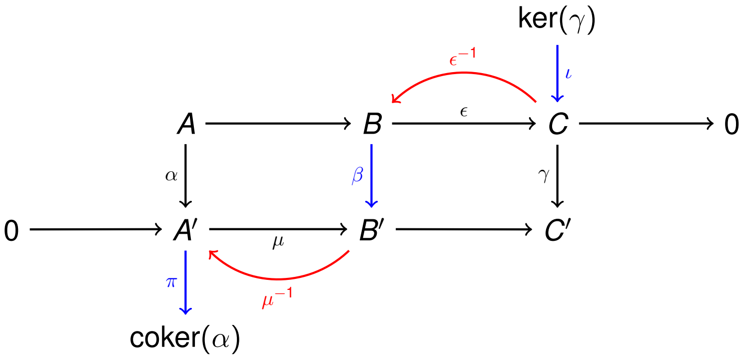
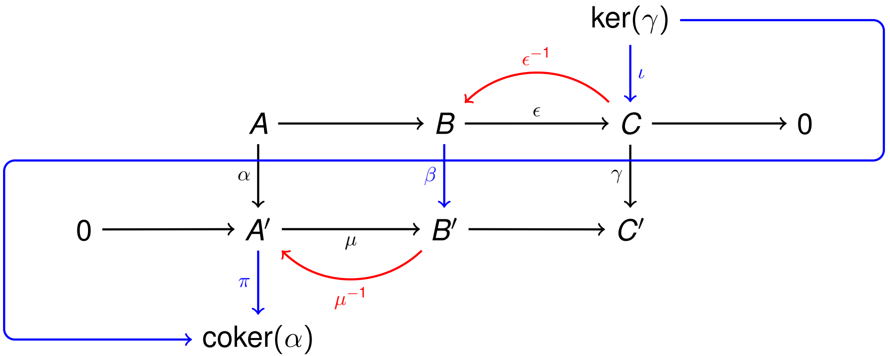


$$\pi \circ \mu^{-1} \circ \beta \circ \epsilon^{-1} \circ \iota$$

Snake Lemma Revisited



∂ is a composition of generalized morphisms!

Generalized Morphisms in CAP

Realization in CAP

- Generalized morphisms are implemented in CAP

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- They are used for:
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Section 2

Flavor of categorical programming in CAP

Intersection of subobjects

Let M be an object and $N_1 \hookrightarrow M$, $N_2 \hookrightarrow M$ subobjects.

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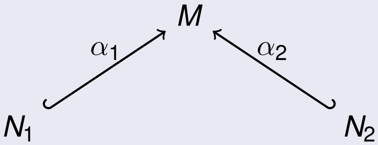

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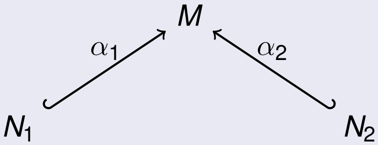

$\mathrm{FiberProduct}(\alpha_1, \alpha_2)$

1. Compute the fiber product of α_1 and α_2.

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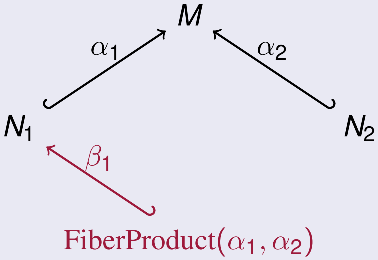

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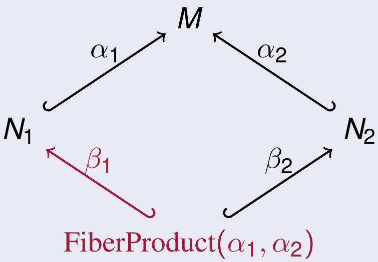

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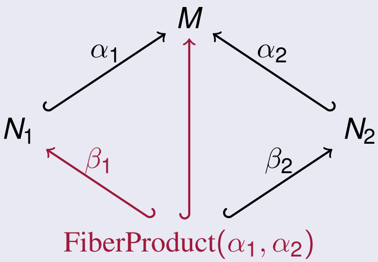

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2. Compute the projection β_1.
3. Return the composition $\alpha_1 \circ \beta_1$.

Fiber product computed by basic algorithms

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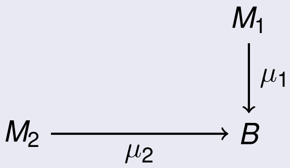

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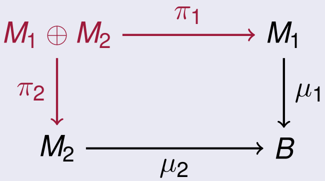

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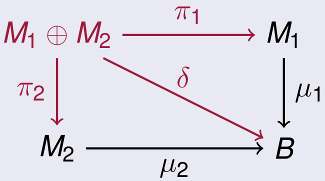

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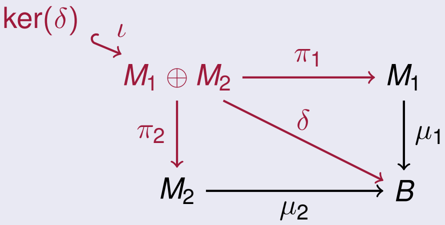

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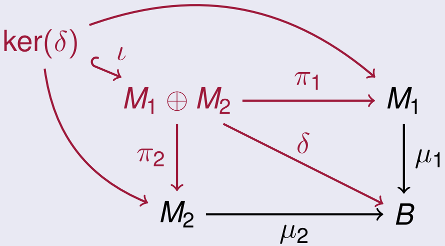

1. Compute $M_1 \oplus M_2$ and projection maps.
2. Compute $\delta := \mu_1 \circ \pi_1 - \mu_2 \circ \pi_2$.
3. Compute the kernel embedding ι of δ, $\pi_1 \circ \iota$, and $\pi_2 \circ \iota$.

Kernel

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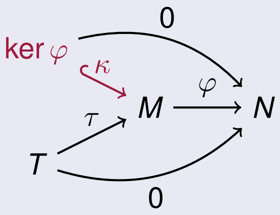

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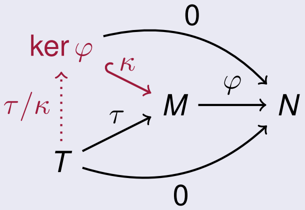
a morphism given by kerφ's universal property.



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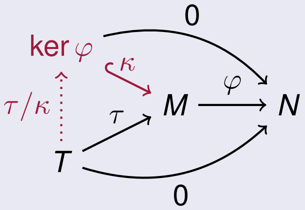

Thus a proper implementation of the kernel needs **three** algorithms.

Basic operations

Build up your algorithms from basic categorical operations:

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Example: Basic operations for fiber product

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- Modules $\vdash$ Gröbner basis computation

Available Constructions

Once all basic operations for a category **A** are implemented,

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Available Constructions

Once all basic operations for a category $\mathbf{A}$ are implemented, CAP can (among other) create

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- the categories of filtered objects of **A**
- and combinations of those!

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Using the basic operations and the constructions described, CAP can compute

- spectral sequences,
- diagram chases,
- natural isomorphisms,
- and much more!

Stacking of data structures

Example: Snake lemma in coherent sheaves over $\mathbb{P}^n$

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\downarrow \\
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$$\downarrow$$

$$G\left(S-\text{grmod}\right)$$

$$\downarrow$$

$$S-\text{grmod}$$

$$\downarrow$$

$$\text{Projectives}(S)$$

Download CAP

CAP is currently not deposited with GAP. You can download it from GitHub:

`https://github.com/homalg-project/CAP_project`