Software Development Within the SPP1489

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TU Kaiserslautern

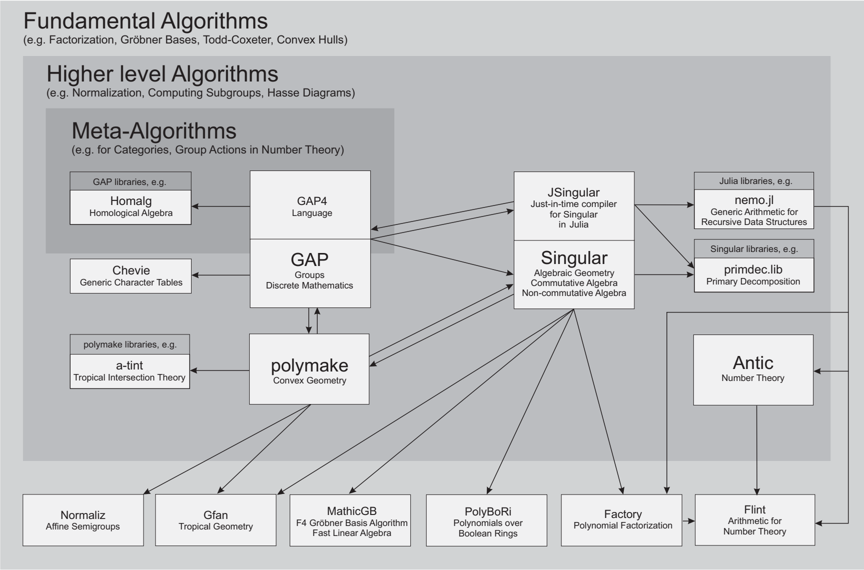
Osnabrück, September 29, 2015

Algorithmic and Experimental Methods
in Algebra, Geometry, and Number Theory
DFG Priority Project SPP 1489

Some basic statements

- Computer algebra algorithms may be implemented in specialized libraries or packages, but it is their incorporation into computer algebra systems - with convenient languages for direct user interaction, comfortable help functions and comprehensive manuals - which make the resulting tools accessible to the interested researcher.

Our Vision


Fundamental Algorithms
(e.g. Factorization, Gröbner Bases, Todd-Coxeter, Convex Hulls)
Higher level Algorithms
(e.g. Normalization, Computing Subgroups, Hasse Diagrams)
Meta-Algorithms
(e.g. for Categories, Group Actions in Number Theory)
GAP libraries, e.g.
Homalg
Homological Algebra
GAP4
Language
GAP
Groups
Discrete Mathematics
Chevie
Generic Character Tables
polymake libraries, e.g.
a-tint
Tropical Intersection Theory
polymake
Convex Geometry
JSingular
Just-in-time compiler for Singular in Julia
Singular
Algebraic Geometry
Commutative Algebra
Non-commutative Algebra
Julia libraries, e.g.
nemo.jl
Generic Arithmetic for Recursive Data Structures
Singular libraries, e.g.
primdec.lib
Primary Decomposition
Antic
Number Theory
Normaliz
Affine Semigroups
Gfan
Tropical Geometry
MathicGB
F4 Gröbner Basis Algorithm
Fast Linear Algebra
PolyBoRi
Polynomials over Boolean Rings
Factory
Polynomial Factorization
Flint
Arithmetic for Number Theory

Some basic statements

- Through computer algebra software, a large treasure of mathematical knowledge becomes accessible to and can also be applied by non-experts.

Citation Record (data from https://zbmath.org)

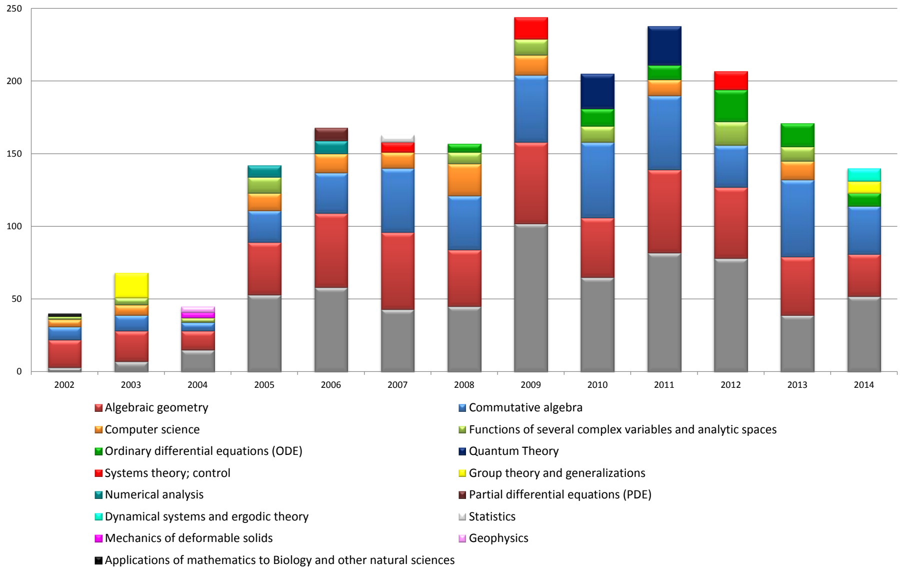
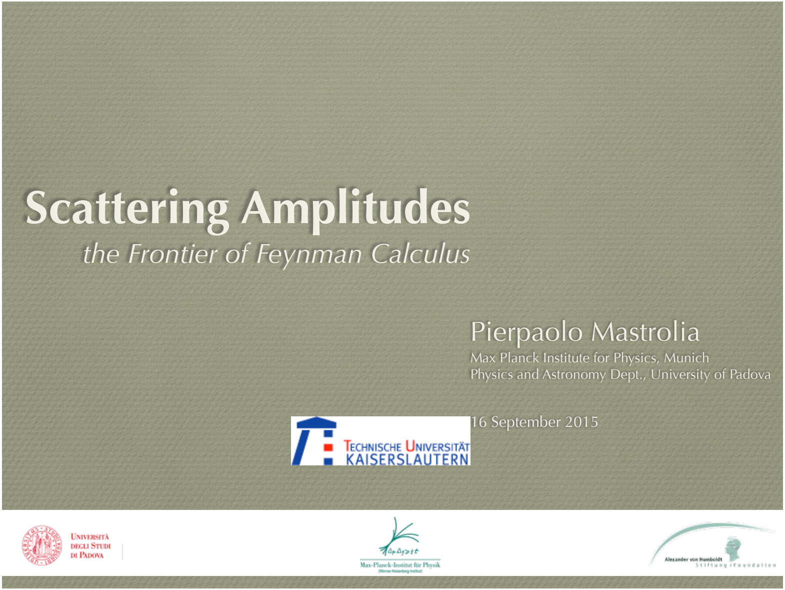


Figure: Top five MSC areas for SINGULAR per year, remaining areas in grey

Some basic statements

- The design and further development of successful computer algebra software is always driven by intended applications, irrespective of whether these applications lie within or outside of mathematics.

Potential Applications: High Energy Physics



Potential Applications: High Energy Physics

High Energy Particle Physics

- Discovery of the Higgs boson: **Standard Model** completed!

We've got a beautiful theory for describing (many, if not all) phenomena at colliders and elsewhere

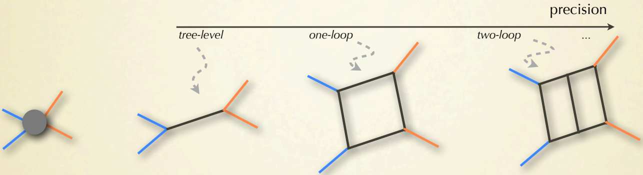
? Higgs properties

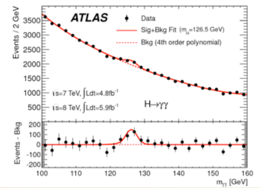
- Higgs couplings extracted from normalization of cross-sections that are sensitive to radiative corrections
- Further verification of Higgs mechanism will require detailed theoretical predictions for production cross-sections and decay rates

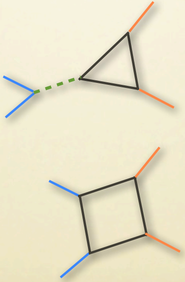
? The Standard Model cannot be the ultimate theory:
(ex. neutrino masses and dark matter not accounted for)
searching for ***New Physics***

- ***New Physics*** effect carried by **massive particles**
- Current major focus (*main stream*):
improving perturbative prediction for partonic cross sections:
a very important and active field of research

Scattering ~ Feynman Diagrams





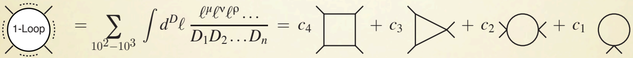


- same signature in the experiment
- only one of them contributes to the little bump
- precision calculation to distinguish them:

  signal vs background analysis

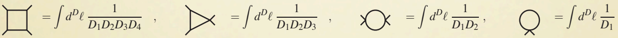
One-Loop Scattering Amplitudes

- n-particle Scattering: $1+2 \to 3+4+\ldots+n$
- Reduction to a Scalar-Integral Basis   Passarino-Veltman



$$= \sum_{10^2-10^3} \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots D_n} = c_4 \,\square + c_3 \,\triangleright + c_2 \,\bigcirc + c_1 \,\bigcirc$$

- Known: Master Integrals



$$\square = \int d^D\ell \, \frac{1}{D_1 D_2 D_3 D_4}, \quad \triangleright = \int d^D\ell \, \frac{1}{D_1 D_2 D_3}, \quad \bigcirc = \int d^D\ell \, \frac{1}{D_1 D_2}, \quad \bigcirc = \int d^D\ell \, \frac{1}{D_1}$$

- Unknowns: c_i are rational functions of external kinematic invariants

Potential Applications: High Energy Physics

Example

```
> ring R = (0,p11,p12,p22,e34,m1,m2,m3), (x1,x2,x3,x4), dp;
> poly D1 = 2*x3*x4*e34+x1*(p11*x1+p12*x2)
.              +(x1*p12+p22*x2)*x2-m1;
> poly D2 = -m2+2*x3*x4*e34-2*p11*x1+p11+x1*(p11*x1+p12*x2)
.              +(x1*p12+p22*x2)*x2-2*p12*x2;
> poly D3 = 2*x3*x4*e34+2*x1*p12-m3+x1*(p11*x1+p12*x2)+p22
.              +(x1*p12+p22*x2)*x2+2*p22*x2;
> ideal I = D1, D2, D3;
> ideal GI = groebner(I);
```

Joint project with Pierpaolo Mastrolia, Tiziano Peraro, and Janko Böhm.

Potential Applications: Surfaces of General Type With $p_g = 0$

"Can a computer classify all surfaces of general type with $p_g = 0$?"

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Work by Castelnuovo, Enriques, Godeaux, Campedelli, Miyaoka, Reid and students, Beauville, Bauer-Catanese and students, and many more.

Potential Applications: Surfaces of General Type With $p_g = 0$

Example (The case $K^2 = 1$: Numerical Godeaux surfaces)

For such a surface X, it is known that $H_1(X, \mathbb{Z})$ is cyclic of order at most 5, and constructions have been given for each choice of order.

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Potential Applications: Surfaces of General Type With $p_g = 0$

Idea of a construction in case $H_1(X, \mathbb{Z}) = 0$

Let $\{x_0, x_1\}$ be a basis of $|2K_S|$ and $\{y_0, y_1, y_2, y_3\}$ a basis of $|3K_S|$.

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Let $\{x_0, x_1\}$ be a basis of $|2K_S|$ and $\{y_0, y_1, y_2, y_3\}$ a basis of $|3K_S|$. We consider the canonical ring $R(S) = \bigoplus_{n \geq 0} \mathrm{H}^0(S, \mathcal{O}_S(nK_S))$ as a module over the weighted polynomial ring $\mathbb{C}[x_0, x_1, y_0, y_1, y_2, y_3]$ and study the image of $X = \mathbf{Proj}(R(S))$ in $\mathbb{P}(2^2, 3^4)$ under the map induced by $|2K_S, 3K_S|$.

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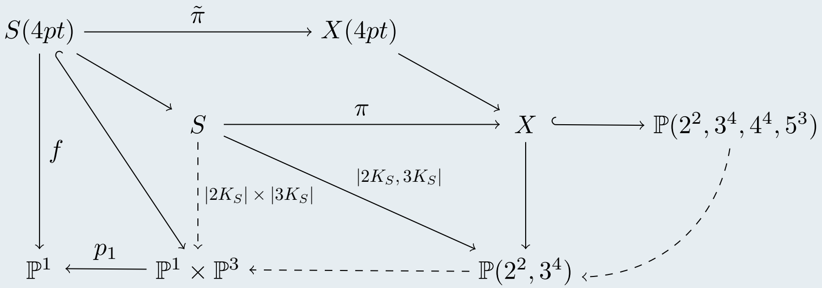
Idea of a construction in case $H_1(X, \mathbb{Z}) = 0$

The bicanonical system $|2K_S|$ has no fixed part and 4 distinct base points.



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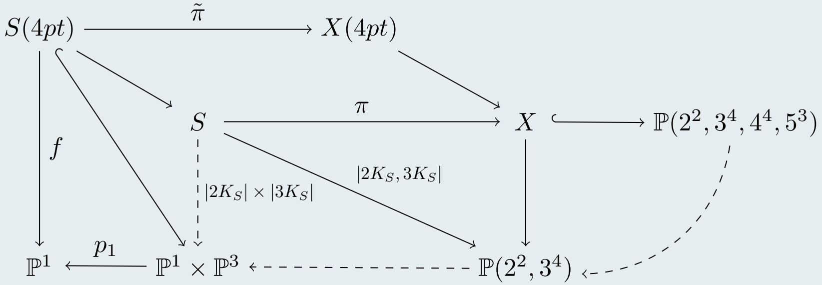

Joint project with Frank-Olaf Schreyer and Isabel Stenger.

Potential Applications: Surfaces of General Type With $p_g = 0$

Why is this computationally hard?

```
> Rextension;
//   characteristic : 0
//   1 parameter    : a
//   minpoly        :
(248738794738328172995583944749904330252605378584297O0*a^8
+412197480758832021377448558823165698794277118212212070*a^7
+625366891611244986389942014312773193649951168354090190*a^6
-436561073546512334083477547357856090524552855592558795*a^5
-914947642504230095779800456657440020138074539186145912*a^4
-2227325279423247966617649640155997715235288113299887954*a^3
+2312070077580715288467637707530192772778088469836344950*a^2
+1366053134215201364075122803745127996518986576734818612*a
-1156759915557562158859054495379551857229358735237021536)
//   number of vars : 12
```

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- normalization: algorithms of de Jong, Greuel-Laplagne-Seelisch.
- means to analyse singularities: Hamburger-Noether expansions, blow-ups, resolution of singularities, and more.

Example

- Dereje Kifle Boku, Wolfram Decker, Claus Fieker, Andreas Steenpass: *Gröbner Bases over Algebraic Number Fields.* Accepted paper for PASCO 2015

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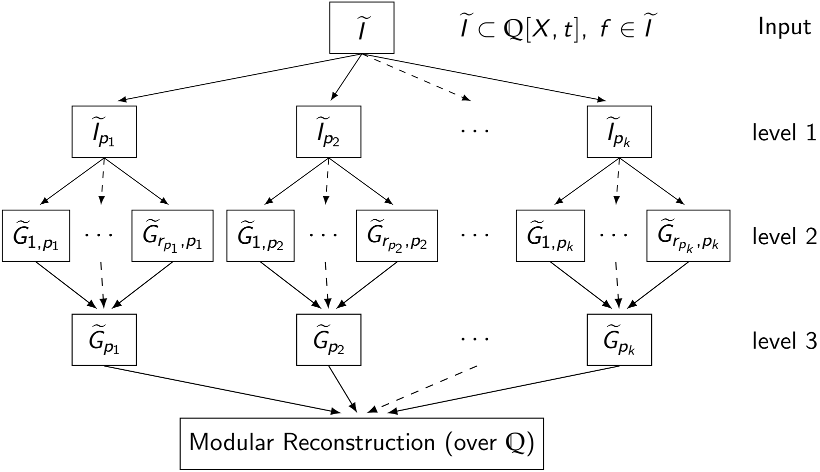
  Gröbner Bases over Algebraic Number Fields.
  Accepted paper for PASCO 2015
- Burcin Erocal, Oleksandr Motsak, Frank-Olaf Schreyer, Andreas Steenpass:
  Refined Algorithms to Compute Syzygies.
  To appear in J. Symb. Comp.

Ongoing work: Gröbner bases over rational function fields, various approaches to computing syzygies.

Example: Gröbner Bases over Number Fields



See talk by Andreas Steenpass.

Second Challenge: Parallelization

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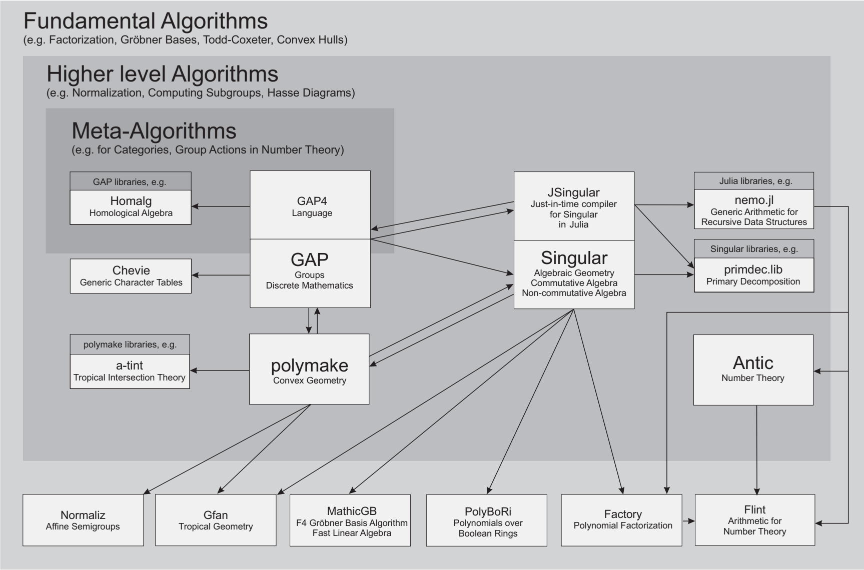
The latter requires, for example, to make the memory managment of the CAS thread-safe in an effective way.

Our Vision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Example

```
> LIB("parallel.lib","random.lib");
> ring R = 0,x(1..4),dp;
> ideal I = randomid(maxideal(3),3,100);
```

Coarse Grained Parallelism in SINGULAR by Example

Example

```
> LIB("parallel.lib","random.lib");
> ring R = 0,x(1..4),dp;
> ideal I = randomid(maxideal(3),3,100);
> proc sizeStd(ideal I, string monord){
      def R = basering; list RL = ringlist(R);
      RL[3][1][1] = monord; def S = ring(RL); setring(S);
      return(size(std(imap(R,I))));}
```

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> list commands = "sizeStd","sizeStd";
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> list commands = "sizeStd","sizeStd";
> list args = list(I,"lp"),list(I,"dp");
> parallelWaitFirst(commands, args);
  [1] empty list
  [2] 11
```

Coarse Grained Parallelism in Singular by Example

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      return(size(std(imap(R,I))));}
> list commands = "sizeStd","sizeStd";
> list args = list(I,"lp"),list(I,"dp");
> parallelWaitFirst(commands, args);
  [1] empty list
  [2] 11
> parallelWaitAll(commands, args);
  [1] 55
  [2] 11
```

Mathematical point of view

There are algorithms whose basic strategy is inherently parallel, whereas others are sequential in nature.

Second Challenge: Parallelization

Mathematical point of view

There are algorithms whose basic strategy is inherently parallel, whereas others are sequential in nature. A prominent example of the former type is Villamayor's constructive version of Hironaka's desingularization theorem.

Example: Resolution of Singularities

Theorem (Hironaka, 1964)

For every algebraic variety over a field K with $\operatorname{char} K = 0$ *a desingularization can be obtained by a finite sequence of blow-ups along smooth centers.*

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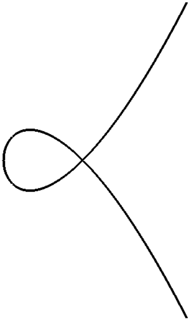
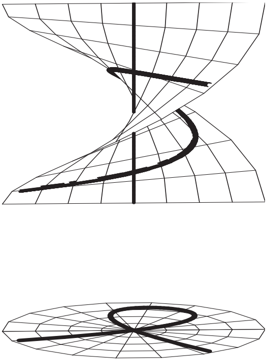
For example, resolve the

node



by one blow-up



which replaces the singular point inside the plane by a line, hence

Example: Resolution of Singularities

Working with blow-ups means to work with different charts.

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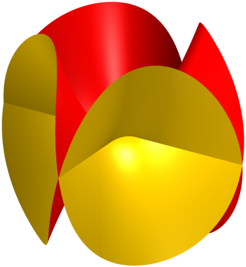
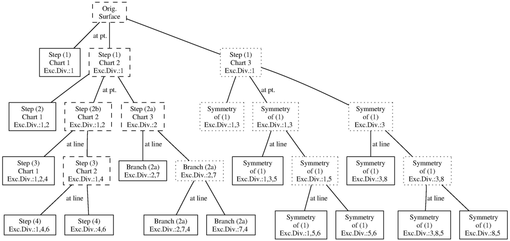
In this way, the resolution of singularities leads to a tree of charts. Here is the graph for resolving the singularities of $z^2 - x^2y^2 = 0$





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$$A = A^{(0)} \subset A^{(1)} \cdots \subset A^{(m)} = \overline{A}.$$

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Example: Local-to-Global Approach to Normalization

New local-to-global approach to normalization

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Janko Boehm, Wolfram Decker, Santiago Laplagne, Gerhard Pfister, Andreas Steenpass, Stefan Steidel:
Parallel algorithms for normalization.
J. Symb. Comp. 51 (2013), 99-114.

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- Stratify the singular locus;

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Third Challenge: Make More and More of the Abstract Concepts of Algebra, Geometry, and Number Theory Constructive

Example: Cohomology

Typical applications of Gröbner Bases and Syzygies in the non-commutative case:

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Use the Weyl algebra to compute the de Rham cohomology of complements of affine varieties. Algorithm by Uli Walther, implemented by Cornelia Rottner in SINGULAR.

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Example (Sheaf cohomology)

Use the exterior algebra to compute the cohomology of coherent sheaves on projective space via the constructive version of the Bernstein-Gel'fand-Gel'fand (BGG) correspondence by Eisenbud-Fløystad-Schreyer.

Example: de Rham Cohomology

Example

```
> ring R = 0, (x,y,z), dp;
> list L = (xy,xz);
> deRhamCohomology(L);
 [1]:
    1
 [2]:
    1
 [3]:
    0
 [4]:
    1
 [5]:
    1
```

Notation

Let V be a vector space of dimension $n+1$ over a field K with dual space $W = V^*$, $S = \mathrm{Sym}_K(W)$ and $E = \bigwedge V$. We grade S and E by taking elements of W to have degree 1, and elements of V to have degree -1.
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The BGG correspondence relates bounded complexes of coherent sheaves on $\mathbb{P}^n$ and minimal doubly infinite free resolutions over E. In particular, it associates to each finitely generated graded S-module a so-called *Tate resolution* which only depends on the sheafification $\widetilde{M}$ and which "reflects" the cohomology of $\widetilde{M}$ and all its twists.

Potential Applications: Hartshorne's Conjecture

"The question of the existence of non-trivial rank 2 vector bundles on $\mathbb{P}^n$, $n \geq 5$, is the most interesting unsolved problem in projective geometry that I know of."

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Conjecture (Hartshorne, 1974)

If Y is a non-singular subvariety of dimension r of $\mathbb{P}^n$, and if $r > \frac{2}{3}n$, then Y is a complete intersection.

The border case

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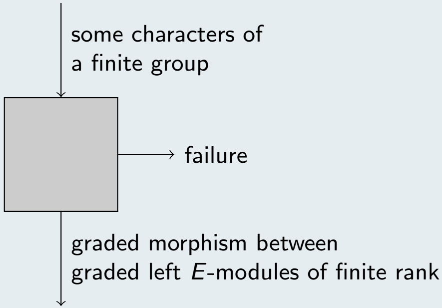
Mohamed Barakat's dream:



A homalg Example: From Groups to Vector Bundles

```
gap> LoadPackage( "repsn" );;
gap> LoadPackage( "GradedModules" );;
gap> G := SmallGroup( 1000, 93 );
<pc group of size 1000 with 6 generators>
gap> Display( StructureDescription( G ) );
((C5 x C5) : C5) : Q8

gap> V := Irr( G )[6];; Degree( V );
5
gap> T0 := Irr( G )[5];; Degree( T0 );
2
gap> T1 := Irr( G )[8];; Degree( T1 );
5
gap> mu0 := ConstructTateMap( V, T0, T1, 2 );
<A homomorphism of graded left modules>
```

A homalg Example:: From groups to Vector Bundles

```
gap> A := HomalgRing( mu0 );
Q{e0,e1,e2,e3,e4}
(weights: [ -1, -1, -1, -1, -1 ])
gap> M:=GuessModuleOfGlobalSectionsFromATateMap(2, mu0);;
gap> ByASmallerPresentation( M );
<A graded non-zero module presented by 92
  relations for 19 generators>

gap> S := HomalgRing( M );
Q[x0,x1,x2,x3,x4]
(weights: [ 1, 1, 1, 1, 1 ])
gap> ChernPolynomial( M );
( 2 | 1-h+4*h^2 ) -> P^4
gap> tate := TateResolution( M, -5, 5 );;
```

A homalg Example:: From Groups to Vector Bundles

```
gap> Display( BettiTable( tate ) );
total:  100  37  14  10   5   2   5  10  14  37 100   ?   ?   ?   ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
    4:  100  35   4   .   .   .   .   .   .   .   .   0   0   0   0
    3:    *   .   2  10  10   5   .   .   .   .   .   .   0   0   0
    2:    *   *   .   .   .   .   .   2   .   .   .   .   .   0   0
    1:    *   *   *   .   .   .   .   .   .   5  10  10   2   .   0
    0:    *   *   *   *   .   .   .   .   .   .   .   .   4  35 100
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---S
twist:   -9  -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
---------------------------------------------------------------------
Euler:  100  35   2 -10 -10  -5   0   2   0  -5 -10 -10   2  35 100
```

Third Challenge: Make More and More of the Abstract Concepts of Algebraic Geometry Constructive

Exploit derived equivalences in computer algebra

Capturing the intrinsic structure of geometric objects in terms of numbers or algebraic objects is a guiding theme in algebraic geometry. With the rapid increase of abstraction initiated by Grothendieck, this relies on more and more involved mathematical language and formalism.

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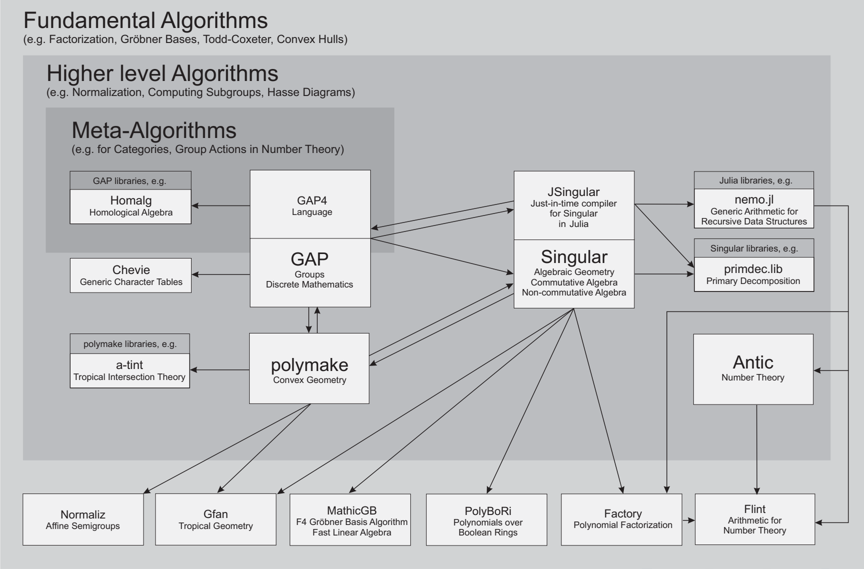
Fourth Challenge: Integration and Interaction of the Computer Algebra Systems and Libraries Involved



Computing the GIT-fan

$I \subset K[x_1, \dots, x_n]$ homogeneous w.r.t. $Q = (q_{ij}) \in \mathbb{Q}^{r \times n}$ and $X = V(I)$.

Another Example: Using POLYMAKE in SINGULAR

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$$Q \text{ defines a torus action} \qquad \mathbb{T}^r \times X \to X$$

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- Uses **polyhedral geometry** (faces, intersection of cones).
- Uses **Gröbner bases** (monomial containment test).
- Action of a **finite symmetry group** makes complicated computations possible.
- Implemented in SINGULAR by Janko Böhm, Simon Keicher, and Yue Ren.

Application: Computing the GIT-fan

We compute the GIT-fan of $\mathbb{G}(2,4)$ in SINGULAR:

Example

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> LIB "gitfan.lib";
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> fan F = gitFan(I, Q);
> rays(F);
  0 0 1
  0 1 0
  0 1 1
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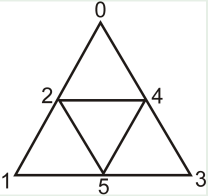
```



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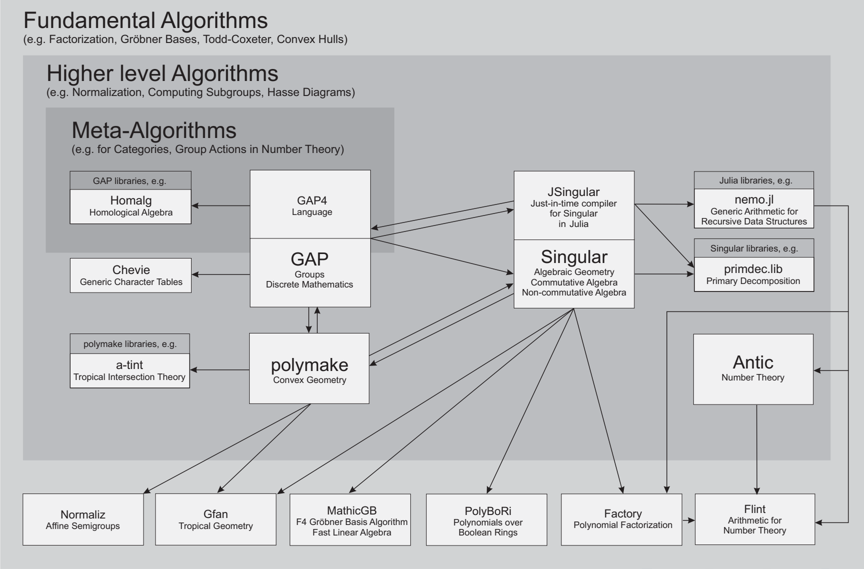

Software for Number Theory

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- Polynomial factorisation over numerous rings.
- Real/complex arithmetic with guaranteed precision via FLINT/ARB extension library.

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Resultant benchmark: Nemo

```
R = GF(17^11)
S = R[y]
T = S/(y^3 + 3x*y + 1)
U = T[z]
f = T(3y^2 + y + x)*z^2 + T((x + 2)*y^2 + x + 1)*z + T(4x*y + 3)
g = T(7y^2 - y + 2x + 7)*z^2 + T(3y^2 + 4x + 1)*z + T((2x + 1)*y + 1)
s = f^12
t = (s + g)^12
time r = resultant(s, t)
```

This benchmark is designed to test generics and computation of the resultant.

SageMath 6.8	Magma V2.21-4	Nemo-0.3
179907s	82s	2s

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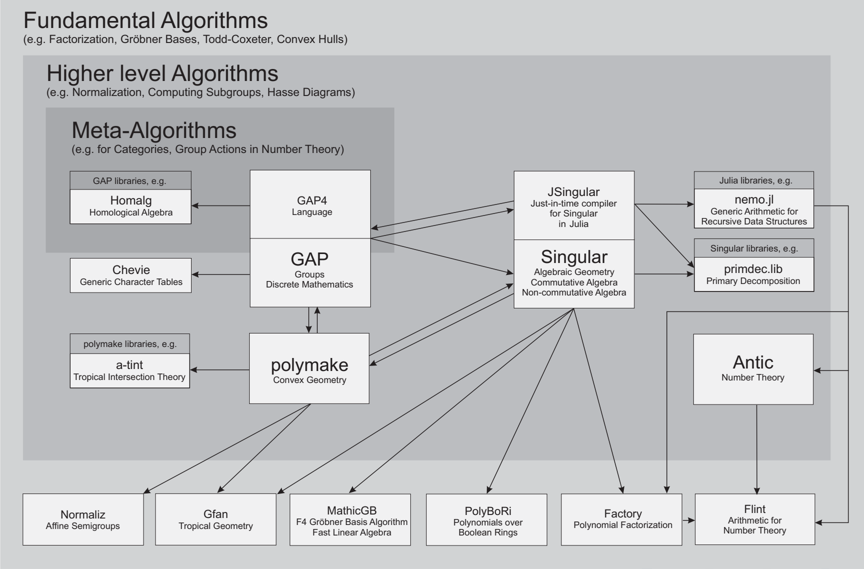
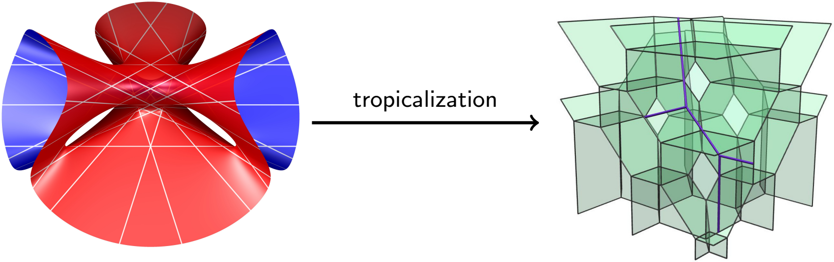


Software for Tropical Geometry

What is tropical geometry?

- Tropical geometry is a piece-vise linear version of algebraic geometry.
- Algebraic objects become discrete / polyhedral objects. Tropical varieties are polyhedral complexes.



GFAN (Anders Jensen) and SINGULAR

1. GFAN
   - first system capable of computing tropical varieties, based on
     - the work of Fukuda-Jensen-Thomas on computing Gröbner fans;
     - the work of Bogart-Jensen-Speyer-Sturmfels-Thomas on computing tropical varieties;
   - algorithms for $\mathbb{Q}$ with trivial valuation ($\rightarrow$ polyhedral fans).
2. SINGULAR
   - algorithms for $\mathbb{Q}$ with p-adic valuation ($\rightarrow$ polyhedral complexes);
   - use commutative algebra, based on the work of Thomas Markwig-Yue Ren, to lift to the trivial valuation case.

See talk by Thomas Markwig.

A-TINT (Simon Hampe)

- There is a notion of tropical intersection theory (Mikhalkin, Allermann-Rau, Francois-Rau, Shaw) on *smooth* tropical varieties.

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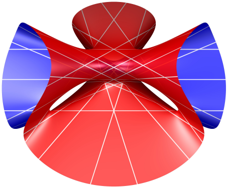
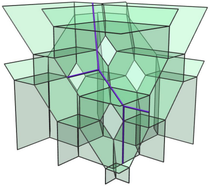
What is A-TINT?

- It is an extension of POLYMAKE (soon to be bundled with POLYMAKE!) for tropical intersection theory.
- Features include: Intersection products for the tropical torus and for smooth surfaces (the latter based on algorithms by Dennis Diefenbach and Kristin Shaw), divisors of rational functions, matroidal fans, moduli spaces of rational curves (including Hurwitz cycles),...
- Webpage: `https://github.com/simonhampe/atint`

Example: Classical Intersection Theory





Example

```
> LIB"schubert.lib";
> variety G = Grassmannian(2,4);
> def R = G.baseRing; setring R;
> sheaf S = makeSheaf(G,subBundle);
> sheaf B = dualSheaf(S)^3;
> integral(G,topChernClass(B));
```

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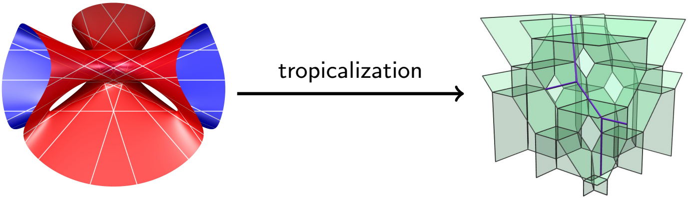

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New EU project in this this direction: OpenDreamKit.

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