Better triangulations in Normaliz 3.0

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- Open source software (GPL)
- developed by W. Bruns, B. Ichim, T. Römer, R. Sieg, and C.S.
- written in C++ (using Boost and GMP/MPIR)
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Applications in: commutative algebra, toric geometry, combinatorics, integer programming, invariant theory, elimination theory, mathematical logic, algebraic topology and theoretical physics.

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Offspring NmzIntegrate computes weighted Ehrhart series and integrals of polynomials over rational polytopes.

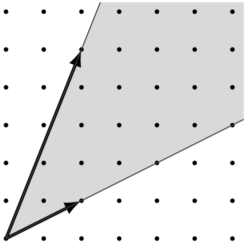
Rational cones

Definition

A lattice L is a subgroup of $\mathbb{Z}^d$. A (rational polyhedral) cone C is a subset

$$\begin{aligned} C &= \operatorname{cone}(x_1, \ldots, x_n) \\ &= \{a_1 x_1 + \cdots + a_n x_n : a_1, \ldots, a_n \in \mathbb{R}_+\} \end{aligned}$$

with a generating system $x_1, \ldots, x_n \in \mathbb{Z}^d$.



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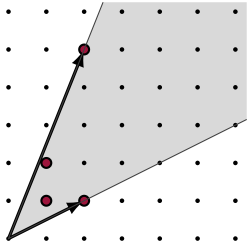
Theorem (Gordan's lemma)

Let $C \subset \mathbb{R}^d$ be the cone generated by $x_1, \dots, x_n \in \mathbb{Z}^d$. Then $C \cap L$ is an affine monoid M, i.e. a finitely generated submonoid of $\mathbb{Z}^d$.



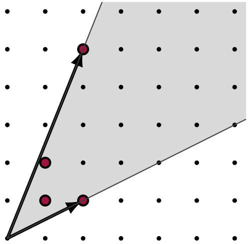
The tasks of Normaliz: Hilbert basis

From now on we assume that C is a pointed cone, i.e.

$$x, -x \in C \implies x = 0.$$

A lattice point $x \in M = C \cap L, x \neq 0$ is irreducible if

$$x = y + z \implies y = 0 \text{ or } z = 0.$$



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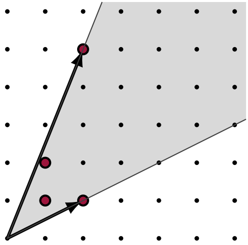


Theorem

There are only finitely many irreducible elements in $C \cap L$ and they form the unique minimal system of generators, the Hilbert basis.

The tasks of Normaliz: Hilbert series

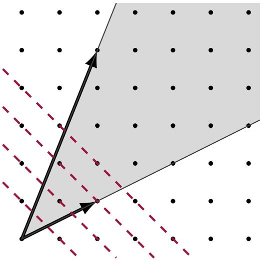
The second main task is to count the lattice points by degree.

The Hilbert (Ehrhart) function is given by

$$H(M, k) = \#\{x \in M : \deg x = k\}$$

and the Hilbert (Ehrhart) series is

$$H_M(t) = \sum_{k=0}^{\infty} H(M, k) t^k.$$



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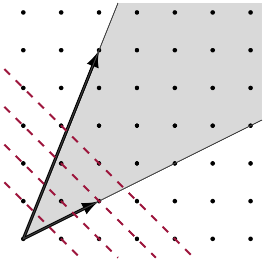
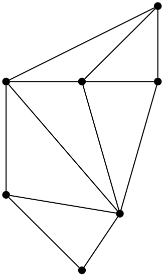


Theorem (Hilbert-Serre, Ehrhart)

- *$H_M(t)$ is a rational function.*
- *$H(M,k)$ is a quasi-polynomial for $k \geq 0$.*

Normaliz Algorithm



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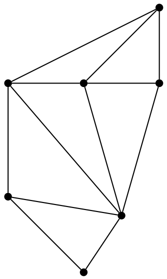

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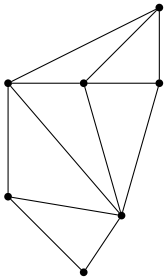

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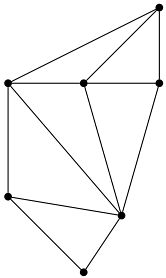

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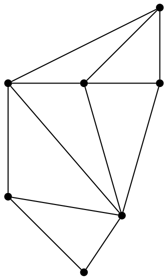

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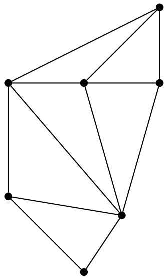

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The two points in blue are the main steps that require the most time.

Recent developments

Recent developments, available in Normaliz 3.0:

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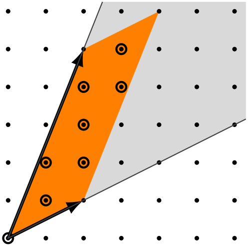
- massive parallelization with Xeon Phi cards
- algorithmic improvements for the computation of the fixed lexicographic triangulation: pyramid decomposition and partial triangulations
- algorithms that allow us to find and use "better" triangulations

Simplicial cones

Let $x_1, \ldots, x_d$ be linearly independent and $S = \operatorname{cone}(x_1, \ldots, x_d)$. Then

$$E = \{q_1 x_1 + \cdots + q_d x_d : 0 \leq q_i < 1\} \cap \mathbb{Z}^d$$

together with $x_1, \ldots, x_d$ generate the monoid $S \cap \mathbb{Z}^d$.



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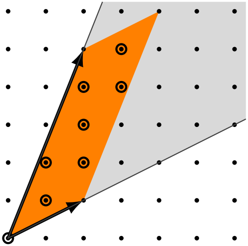

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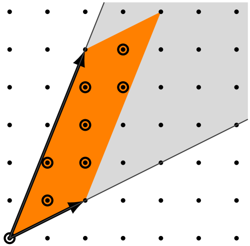

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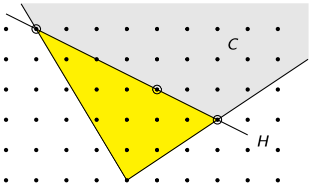
The elements in E are candidates for the Hilbert basis and their number is given by

$$|E| = \det(x_1, \dots, x_d).$$

Therefore the sum of the determinants of the simplices it is a critical size for the runtime of Normaliz.

Bottom decomposition

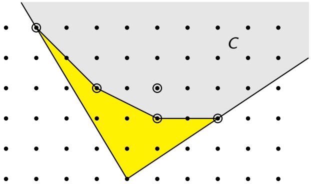
The determinant sum of the triangulation computed by Normaliz depends considerably on the order of the generators of the cone C. Unless they lie in a hyperplane H, then the determinant is exactly the normalized (lattice) volume of the polytope spanned by 0 and $C \cap H$.



This observation helps to find an optimal triangulation in the general case.

Bottom decomposition

We look at the bottom of the polyhedron generated by $x_1, \dots, x_n$ as vertices and C as recession cone, and take the volume underneath the bottom:



With the option `BottomDecomposition`, `-b`, Normaliz 3.0 computes a triangulation that respects the bottom facets. This gives the optimal determinant sum for the given generators.

While bottom decomposition is not used automatically for C, it is used for large simplicial cones in the triangulation if Normaliz can subdivide them.

Ordering of generators

The order of the vectors can play an enormous role.
Normaliz 3.0 orders the input vectors (after coordinate transformation) as follows:

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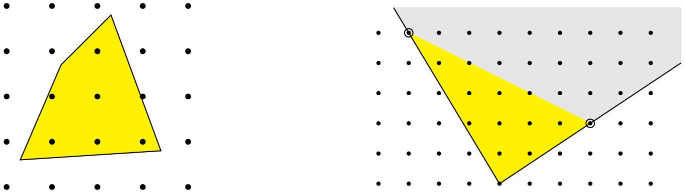
Computation time reductions for the linear ordering polytope for $n = 6$: support hyperplanes: $35s \to 5s$, Hilbert basis: $72s \to 7s$.

Approximation of rational polytopes

Often one wants to compute lattice points in rational polytopes. If the denominators of the vertices are large, a direct application of the Normaliz primal algorithm can easily fail because the determinants of the simplicial cones are enormous.



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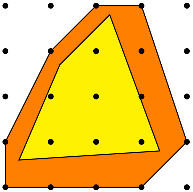

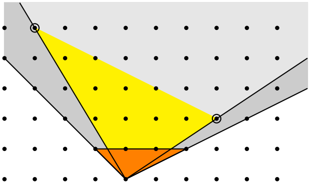

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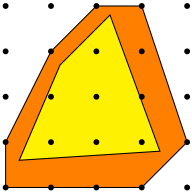

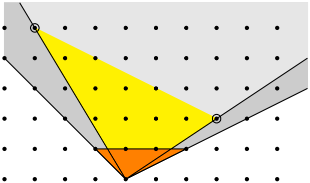

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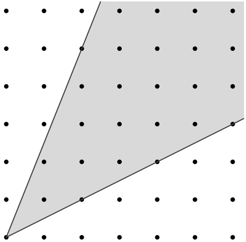
In order to use this method "globally" for P, one uses the option `Approximate`, `-r`. It is not used automatically since it could increase the geometric complexity in an unpredictable way.

Decompose simplicial cones



For a simplex with big volume, we decompose it into smaller simplices such that the sum of their volumes decreases remarkably.
For this purpose we compute points from the cone and use them for a new triangulation.

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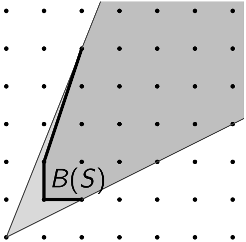

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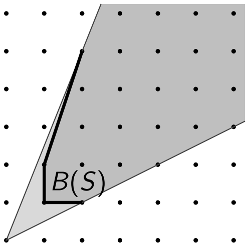

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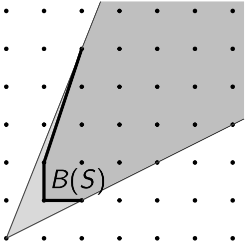

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- the approximation of the simplex, or
- integer programming methods.

Decomposition algorithm

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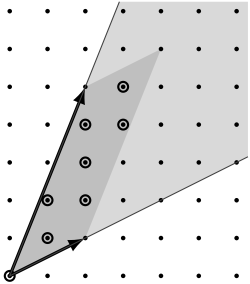

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Decomposition algorithm

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compute a point x that minimizes the sum of determinants:

$$\sum_{i=1}^{d} \det(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_d) = N^T x,$$

where N is a normal vector on the hyperplane spanned by $x_1, \ldots, x_d$.

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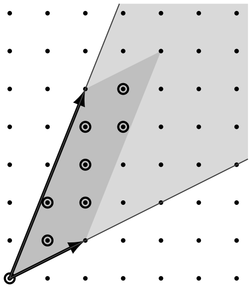

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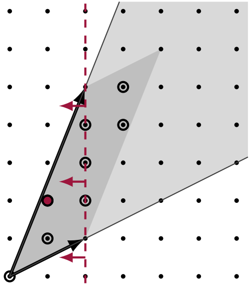

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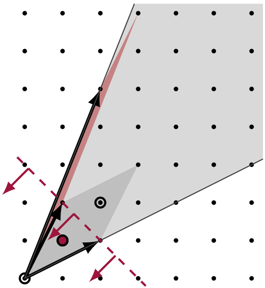

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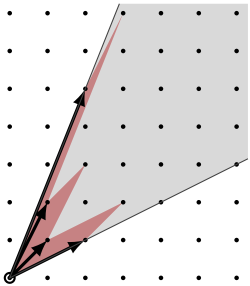
If problem can be solved: form a stellar subdivision with the solution.

Decomposition algorithm

$S = \mathrm{cone}(x_1, \ldots, x_d)$



GOAL

compute a point x that minimizes the sum of determinants:

$$\sum_{i=1}^{d} \det(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_d) = N^T x,$$

where N is a normal vector on the hyperplane spanned by $x_1, \ldots, x_d$.

SOLVE THE IP

$$\min\{N^T x : x \in S \cap \mathbb{Z}^d, x \neq 0, N^T x < N^T x_1\} \qquad (\star)$$

If problem can be solved: form a stellar subdivision with the solution.

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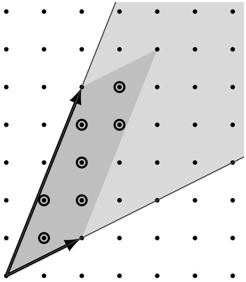
	hickerson-16	hickerson-18	knapsack_11_60
simplex volume	9.83×10^7	4.17×10^{14}	2.8×10^{14}
bottom volume	8.10×10^5	3.86×10^7	2.02×10^7
our volume	3.93×10^6	5.47×10^7	2.39×10^7
old runtime	2s	$>$12d	$>$8d
new runtime	0.5s	46s	5.1s

SUN xFire 4450, 4 Intel Xeon X7460, 20 threads

Decomposition via approximation

Instead of solving the IP, we can approximate the simplicial cone to find possible subdivision elements.

Approximate simplicial cone



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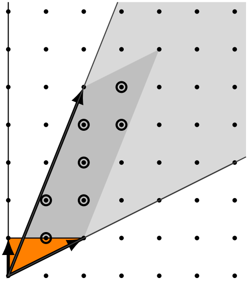

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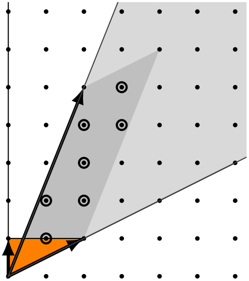
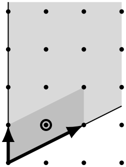


Compute the approximating cone



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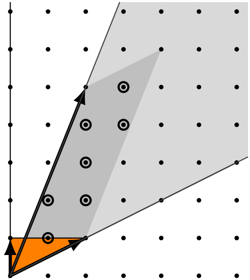

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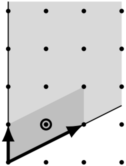

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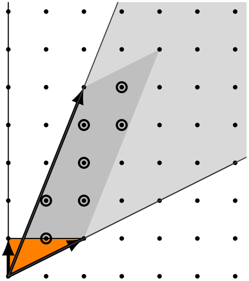

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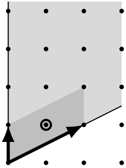
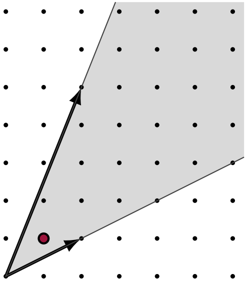


Subdivide simplicial cone



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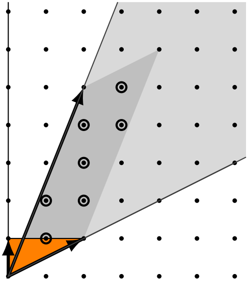

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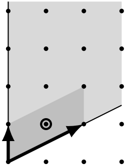

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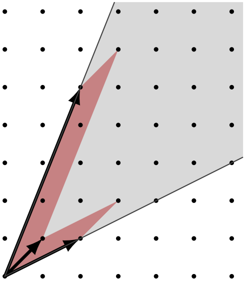

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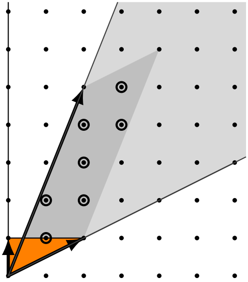

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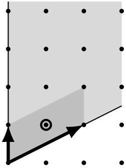

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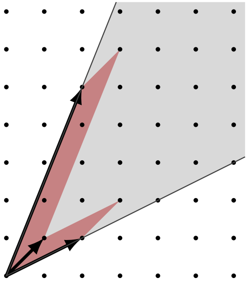

Using SCIP and the approximation might be used in combination and the approximation might be redone in a higher level.

Decomposition via approximation

Note: After subdivision the decomposition of the cone may no longer be a triangulation in the strict sense, but a decomposition that we call a nested triangulation.