How much randomness is needed for high-confidence Monte Carlo integration?

Robert J. Kunsch
RWTH Aachen University
kunsch@mathc.rwth-aachen.de

We study Monte Carlo methods for integrating smooth functions based on n function evaluations.

The classical way of assessing the precision ε of a randomized integration method is based on the *root mean squared error* (RMSE) or the *mean error*. Optimal Monte Carlo error rates in terms of the mean error are well known for classical Sobolev spaces $W_p^r\left([0,1]^d\right)$ and can be achieved with methods like control variates, in some cases also via stratified sampling. For spaces $\mathbf{W}_p^{r,\mathrm{mix}}\left([0,1]^d\right)$ of dominating mixed smoothness, optimal integration methods based on a randomly shifted and dilated Frolov rule have been found in [1, 2]. If, however, the error is measured in terms of small error ε with high probability $1-\delta$, the so-called *probabilistic* error criterion, see [3], some of the aforementioned methods turn out to be suboptimal with the error $\varepsilon = e(n,\delta)$ depending polynomially on δ^{-1} instead of the polynomial dependence on $\log \delta^{-1}$ we hope for. Optimality in classical Sobolev spaces can be restored for control variates employing the median-of-means, for stratified sampling concentration phenomena (Hoeffding's inequality) can lead to optimality; in any event, the amount of random numbers in such optimal methods is proportional to n. The randomized Frolov rule which uses only $2d$ random parameters independently of n, however, turns out to be suboptimal with respect to the probabilistic error criterion.

This raises the question: How small can the probabilistic error be if we limit the amount of randomness? *Restricted* Monte Carlo methods that only use a small amount of random bits have been studied in [4] for the RMSE criterion. A similar study for the probabilistic error criterion of restricted Monte Carlo methods will be presented.

Joint work with: Daniel Rudolf.

References

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